Reducible to linear differential equations

Sometimes a nonlinear DE can be reduced to a linear DE. One set of equations for which this can always be achieved is the class of "***Bernoulli Equation***". These are of the form,

$$\frac{dy}{dx} + P(x).y = Q(x).y^n .$$

When $n = 0$ or $n = 1$, the above DE is linear.

Consider the case when $n \neq 0,1$; division by y^n yields

$$y^{-n}\frac{dy}{dx} + P(x).y^{1-n} = Q(x) .$$

Let $z = y^{1-n}$ $\Rightarrow$ $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ $\Rightarrow$ $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}.\frac{dz}{dx}$,

$$\therefore \frac{1}{1-n}.\frac{dz}{dx} + P(x).z = Q(x) ,$$

or $\dfrac{dz}{dx} + (1-n)P(x).z = (1-n)Q(x)$. (Linear DE with respect to *z*)

So, Bernoulli's equation can be reduced to a linear DE by the transformation,

$$z = y^{1-n}.$$

Example 1: Solve $xdy + (3y - x^3y^2)dx = 0$.

Solution :

$$xdy + (3y - x^3y^2)dx = 0 \quad \Rightarrow \quad \frac{dy}{dx} + \frac{3y - x^3y^2}{x} = 0,$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{3}{x}.y = x^2y^2. \qquad \text{(Bernoulli's equation)}$$

Division by y^2 gives

$$y^{-2}\frac{dy}{dx} + \frac{3}{x}.y^{-1} = x^2.$$

Let $z = y^{-1} \quad \Rightarrow \quad \dfrac{dz}{dx} = -y^{-2}\dfrac{dy}{dx} \quad \Rightarrow \quad y^{-2}\dfrac{dy}{dx} = -\dfrac{dz}{dx}$,

$$\therefore \quad -\frac{dz}{dx} + \frac{3}{x}.z = x^2,$$

$$\Rightarrow \quad \frac{dz}{dx} - \frac{3}{x}.z = -x^2, \qquad \text{(Linear DE with respect to } z\text{)}$$

where, $P(x) = \dfrac{-3}{x}$ and $Q(x) = -x^2$.

$$\mu = e^{\int P(x)dx} \quad \Rightarrow \quad \mu = e^{\int \frac{-3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3} = \frac{1}{x^3}.$$

$$\mu.z = \int \mu.Qdx + C \quad \Rightarrow \quad \frac{1}{x^3}.z = \int \frac{1}{x^3}.(-x^2)dx + C,$$

$$\Rightarrow \quad \frac{z}{x^3} = -\int \frac{1}{x}dx + C \quad \Rightarrow \quad \frac{z}{x^3} = -\ln x + C,$$

$$\Rightarrow \quad z = x^3(C - \ln x). \qquad \text{But} \quad z = y^{-1} = \frac{1}{y},$$

$$\therefore \quad \frac{1}{y} = x^3(C - \ln x) \qquad \text{or} \qquad y = \frac{1}{x^3(C - \ln x)} \qquad \text{(G.S)}$$

Example 2: Solve $x^2dx-(\sin y\cos^2 y+x^3\tan y)dy=0$.

Solution :

$$x^2dx-(\sin y\cos^2 y+x^3\tan y)dy=0 \;\Rightarrow\; \frac{dy}{dx}-\frac{x^2}{\sin y\cos^2 y+x^3\tan y}=0 \quad \text{(Nonlinear)}$$

But $\frac{dx}{dy}-\frac{\sin y\cos^2 y+x^3\tan y}{x^2}=0 \;\Rightarrow\; \frac{dx}{dy}-(\tan y).x=(\sin y\cos^2 y).x^{-2}$ (Bernoulli)

Division by x^{-2} gives $\quad x^2\frac{dx}{dy}-(\tan y).x^3=\sin y\cos^2 y$.

Let $\quad z=x^3 \;\Rightarrow\; \frac{dz}{dy}=3x^2\frac{dx}{dy} \;\Rightarrow\; x^2\frac{dx}{dy}=\frac{1}{3}.\frac{dz}{dy}$,

$$\therefore\; \frac{1}{3}.\frac{dz}{dy}-(\tan y).z=\sin y\cos^2 y \;\Rightarrow\; \frac{dz}{dy}-(3\tan y).z=3\sin y\cos^2 y, \quad \text{(Linear w.r.t z)}$$

where, $\quad P(y)=-3\tan y \quad$ and $\quad Q(y)=3\sin y\cos^2 y$.

$$\mu=e^{\int P(y)dy} \;\Rightarrow\; \mu=e^{\int -3\tan y.dy}=e^{-3\int\frac{\sin y}{\cos y}dy}=e^{3\ln\cos y}=\cos^3 y.$$

$$\mu.z=\int\mu.Qdy+C \;\Rightarrow\; \cos^3 y.z=\int\cos^3 y.(3\sin y\cos^2 y)dy+C,$$

$$\Rightarrow\; \cos^3 y.z=3\int\sin y.\cos^5 y.dy+C \;\Rightarrow\; \cos^3 y.z=-3.\frac{\cos^6 y}{6}+C,$$

but $\quad z=x^3 \;\Rightarrow\; x^3\cos^3 y=-\frac{\cos^6 y}{2}+C$,

or $\quad x^3=C\sec^3 y-\frac{1}{2}\cos^3 y. \qquad$ (G.S)

5- Second order DE reduced to first order DE

***A)** When the dependent variable does not appear in the DE;*

If the dependent variable (say y) does not appear in a second order DE, then this equation can be reduced to a first order DE by letting

$$z=f(x)=\frac{dy}{dx} \;\Rightarrow\; \frac{d}{dx}(z)=\frac{d}{dx}\left(\frac{dy}{dx}\right) \;\Rightarrow\; \frac{dz}{dx}=\frac{d^2y}{dx^2}.$$

Example: Solve $y'' + 2y' = 4x$. (or $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4x$)

Solution :

Since the dependent variable (y) does not appear in the given second order DE, then this equation can be reduced to a first order DE by letting

$$z = f(x) = \frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dz}{dx} = \frac{d^2y}{dx^2}.$$

$\therefore \ \frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4x$ is reduced to

$\frac{dz}{dx} + 2z = 4x$, (Linear 1st order DE w.r.t z)

where, $P(x) = 2$ and $Q(x) = 4x$.

$$\mu = e^{\int P(x)dx} \qquad \Rightarrow \qquad \mu = e^{\int 2dx} = e^{2x}.$$

$$\mu.z = \int \mu.Qdx + C \qquad \Rightarrow \qquad e^{2x}.z = \int e^{2x}.(4x)dx + C_1,$$

$$\Rightarrow \qquad ze^{2x} = 4\int xe^{2x}dx + C_1 \qquad \Rightarrow \qquad ze^{2x} = 4[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}] + C_1,$$

$$\Rightarrow \qquad ze^{2x} = 2xe^{2x} - e^{2x} + C_1 \qquad \Rightarrow \qquad z = 2x - 1 + C_1 e^{-2x},$$

but $z = \frac{dy}{dx} \quad \Rightarrow \quad \therefore \ \frac{dy}{dx} = 2x - 1 + C_1 e^{-2x}$, (Separable DE)

$$\Rightarrow \qquad y = x^2 - x - \frac{1}{2}C_1 e^{-2x} + C_2. \qquad \text{(G.S)}$$

***B)** When the independent variable does not appear in the DE;*

If the independent variable (say x) does not appear in a second order DE, then this equation can be reduced to a first order DE by letting

$$z = f(y) = \frac{dy}{dx} \qquad \Rightarrow \qquad \frac{d}{dx}(z) = \frac{d}{dx}(\frac{dy}{dx}) \qquad \Rightarrow \qquad \frac{dz}{dy}.\frac{dy}{dx} = \frac{d^2y}{dx^2},$$

but $z = \frac{dy}{dx} \quad \Rightarrow \quad \therefore \ z.\frac{dz}{dy} = \frac{d^2y}{dx^2}$.

Example: Solve $4y(y')^2 y'' = (y')^4 + 3$.

Solution :

Since the independent variable (x) does not appear in the given second order DE, then this equation can be reduced to a first order DE by letting

$$z = f(y) = \frac{dy}{dx} \quad \Rightarrow \quad \frac{d}{dx}(z) = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad \Rightarrow \quad \frac{dz}{dy}.\frac{dy}{dx} = \frac{d^2y}{dx^2},$$

but $z = \dfrac{dy}{dx} \quad \Rightarrow \quad \therefore \quad z.\dfrac{dz}{dy} = \dfrac{d^2y}{dx^2}$.

$\therefore \quad 4y(y')^2 y'' = (y')^4 + 3$ is reduced to

$$4yz^2\left(z.\frac{dz}{dy}\right) = z^4 + 3. \qquad \text{(Separable DE)}$$

$$\therefore \quad \frac{4z^3dz}{z^4+3} = \frac{dy}{y} \quad \Rightarrow \quad \ln(z^4+3) = \ln y + C \quad \Rightarrow \quad \ln\left(\frac{z^4+3}{y}\right) = C,$$

$$\Rightarrow \quad \frac{z^4+3}{y} = C_1 \quad [\text{where } C_1 = e^C] \quad \Rightarrow \quad z^4 = C_1 y - 3,$$

$\Rightarrow \quad z = (C_1 y - 3)^{1/4}$. But $z = \dfrac{dy}{dx}$,

$\therefore \quad \dfrac{dy}{dx} = (C_1 y - 3)^{1/4}$, (Separable DE)

$$\therefore \quad \frac{dy}{(C_1 y - 3)^{1/4}} = dx \quad \Rightarrow \quad (C_1 y - 3)^{-1/4} dy = dx,$$

$$\Rightarrow \quad \frac{4}{3C_1}(C_1 y - 3)^{3/4} = x + C_2,$$

or $\quad (C_1 y - 3)^{3/4} = \dfrac{3C_1}{4}(x + C_2)$. (G.S)