II) By inspection

Inspection may be used when the integrating factor is simple such that it can be expected. This occurs when the DE can be rewritten as one of the following forms:

Differential expression	Integrating factor	Exact differential
$x.dy + y.dx$	1	$x.dy + ydx = d(xy)$
$x.dy + y.dx$	$\frac{1}{xy}$	$\frac{xdy + ydx}{xy} = d(\ln xy)$
$x.dy + y.dx$	$\frac{1}{x^2y^2}$	$\frac{xdy + ydx}{x^2y^2} = -d(\frac{1}{xy})$
$x.dy - y.dx$	$\frac{1}{x^2}$	$\frac{xdy - ydx}{x^2} = d(\frac{y}{x})$
$x.dy - y.dx$	$\frac{1}{y^2}$	$\frac{xdy - ydx}{y^2} = -d(\frac{x}{y})$
$x.dy - y.dx$	$\frac{1}{xy}$	$\frac{xdy - ydx}{xy} = d(\ln\frac{y}{x}) = -d(\ln\frac{x}{y})$
$x.dy - y.dx$	$\frac{2}{x^2 - y^2}$	$\frac{2xdy - 2ydx}{x^2 - y^2} = d(\ln\frac{x+y}{x-y}) = -d(\ln\frac{x-y}{x+y})$
$x.dy - y.dx$	$\frac{1}{x^2 + y^2}$	$\frac{xdy - ydx}{x^2 + y^2} = d(\tan^{-1}\frac{y}{x}) = -d(\tan^{-1}\frac{x}{y})$
$x.dy - y.dx$	$\frac{1}{x\sqrt{x^2 + y^2}}$	$\frac{xdy - ydx}{x\sqrt{x^2 + y^2}} = d(\sinh^{-1}\frac{y}{x})$
$x.dy - y.dx$	$\frac{1}{y\sqrt{x^2 + y^2}}$	$\frac{xdy - ydx}{y\sqrt{x^2 + y^2}} = -d(\sinh^{-1}\frac{x}{y})$
$x.dx + y.dy$	2	$2xdx + 2ydy = d(x^2 + y^2)$
$x.dx + y.dy$	$\frac{2}{x^2 + y^2}$	$\frac{2xdx + 2ydy}{x^2 + y^2} = d[\ln(x^2 + y^2)]$
$x.dx + y.dy$	$\frac{1}{\sqrt{x^2 + y^2}}$	$\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = d(\sqrt{x^2 + y^2})$
$x.dx - y.dy$	$\frac{1}{\sqrt{x^2 - y^2}}$	$\frac{xdx - ydy}{\sqrt{x^2 - y^2}} = d(\sqrt{x^2 - y^2})$

Example 1: Solve $x.dy + (x^3 - y).dx = 0$.

Solution :

$$x.dy + (x^3 - y).dx = 0 \qquad \Rightarrow \qquad xdy - ydx + x^3dx = 0,$$

$$\Rightarrow \quad \frac{xdy - ydx}{x^2} + xdx = 0 \qquad \Rightarrow \qquad d\left(\frac{y}{x}\right) + xdx = 0,$$

$$\therefore \quad \frac{y}{x} + \frac{x^2}{2} = C_1 \qquad \text{or} \qquad 2y + x^3 = Cx. \qquad [C = 2C_1] \qquad \text{(G.S)}$$

Example 2: Solve $x.dx+(y-\sqrt{x^2+y^2}).dy=0$.

Solution :

$$x.dx+(y-\sqrt{x^2+y^2}).dy=0 \quad \Rightarrow \quad xdx+ydy-\sqrt{x^2+y^2}dy=0,$$

$$\Rightarrow \quad \frac{xdx+ydy}{\sqrt{x^2+y^2}}-dy=0 \quad \Rightarrow \quad d\left(\sqrt{x^2+y^2}\right)-dy=0$$

$$\therefore \quad \sqrt{x^2+y^2}-y=C. \qquad \text{(G.S)}$$

Example 3: Solve $y.dx-(x^2+y^2+x).dy=0$.

Solution :

$$y.dx-(x^2+y^2+x).dy=0 \quad \Rightarrow \quad -xdy+ydx-(x^2+y^2)dy=0,$$

$$\Rightarrow \quad xdy-ydx+(x^2+y^2)dy=0 \quad \Rightarrow \quad \frac{xdy-ydx}{x^2+y^2}+dy=0,$$

$$\Rightarrow \quad d\left(\tan^{-1}\frac{y}{x}\right)+dy=0 \quad \Rightarrow \quad \tan^{-1}\frac{y}{x}+y=C,$$

$$\text{or} \quad \tan^{-1}\frac{y}{x}=C-y \quad \Rightarrow \quad \frac{y}{x}=\tan(C-y) \quad \Rightarrow \quad x=\frac{y}{\tan(C-y)}. \qquad \text{(G.S)}$$

Example 4: Solve $(xy^2+y).dx+x.dy=0$.

Solution :

$$(xy^2+y).dx+x.dy=0 \quad \Rightarrow \quad xy^2dx+xdy+ydx=0,$$

$$\Rightarrow \quad \frac{dx}{x}+\frac{xdy+ydx}{x^2y^2}=0 \quad \Rightarrow \quad \frac{dx}{x}-d\left(\frac{1}{xy}\right)=0,$$

$$\Rightarrow \quad \ln x-\frac{1}{xy}=C. \qquad \text{(G.S)}$$

Example 5: Solve $2y^2dx+(2x+3xy)dy=0$.

Solution :

$$2y^2dx+(2x+3xy)dy=0 \quad \Rightarrow \quad 2y^2dx+3xydy+2xdy=0, \quad (\times \ \ xy)$$

$$\Rightarrow \quad 2xy^3dx + 3x^2y^2dy + 2x^2ydy = 0 \quad \Rightarrow \quad d(x^2y^3) + 2x^2ydy = 0, \quad (\times \frac{1}{x^2y^3})$$

$$\Rightarrow \quad \frac{d(x^2y^3)}{x^2y^3} + \frac{2}{y^2}dy = 0 \qquad \Rightarrow \qquad \ln(x^2y^3) - \frac{2}{y} = C,$$

or $\quad y\ln(x^2y^3) - 2 = Cy.$ (G.S)

4- Linear differential equations

The general form of the first order linear differential equation is

$$\frac{dy}{dx} + P(x).y = Q(x).$$

(Note that if $P(x) = 0$ or $Q(x) = 0$, then the above DE is a separable variables DE).

If the above linear DE is not exact DE, so it can be reduced to exact one by multiplying it with a suitable integrating factor which can be found as follows,

$$\frac{dy}{dx} + P(x).y = Q(x) \quad \Rightarrow \quad dy + P(x).ydx = Q(x)dx \quad \Rightarrow \quad [P(x)y - Q(x)]dx + dy = 0,$$

$$M(x,y) = P(x)y - Q(x) \qquad \Rightarrow \qquad \frac{\partial M}{\partial y} = P(x),$$

$$N(x,y) = 1 \qquad \Rightarrow \qquad \frac{\partial N}{\partial x} = 0.$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{P(x) - 0}{1} = P(x) \quad \text{(function of } x \text{ only)}$$

$$\therefore \mu = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx} \qquad \Rightarrow \qquad \mu = e^{\int P(x)dx}.$$

Multiplying the given DE by the above integrating factor (I.F) gives

$$e^{\int P(x)dx}[P(x)y - Q(x)]dx + e^{\int P(x)dx}dy = 0,$$

$$\Rightarrow \quad e^{\int P(x)dx}.P(x)ydx + e^{\int P(x)dx}dy - Q(x).e^{\int P(x)dx}dx = 0,$$

$$\Rightarrow \quad d\left[e^{\int P(x)dx}.y\right] = Q(x).e^{\int P(x)dx}dx \quad \Rightarrow \quad \therefore \quad e^{\int P(x)dx}.y = \int Q(x).e^{\int P(x)dx}dx + C.$$

Or simply $\qquad \mu.y = \int \mu.Qdx + C.$

Example 1: Solve $(x^2+x)dy=(x^5+3xy+3y)dx$.

Solution :

$$(x^2+x)dy=(x^5+3xy+3y)dx \quad \Rightarrow \quad x(x+1)\frac{dy}{dx}=x^5+3y(x+1),$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{x^4}{x+1}+\frac{3y}{x} \quad \Rightarrow \quad \frac{dy}{dx}-\frac{3}{x}y=\frac{x^4}{x+1}, \quad \text{(Linear DE with respect to } y\text{)}$$

where, $P(x)=\frac{-3}{x}$ and $Q(x)=\frac{x^4}{x+1}$.

$$\mu=e^{\int P(x)dx} \quad \Rightarrow \quad \mu=e^{\int \frac{-3}{x}dx}=e^{-3\ln x}=e^{\ln x^{-3}}=x^{-3}=\frac{1}{x^3}.$$

$$\mu.y=\int \mu.Qdx+C \quad \Rightarrow \quad \frac{1}{x^3}.y=\int \frac{1}{x^3}.\frac{x^4}{x+1}dx+C,$$

$$\Rightarrow \quad \frac{y}{x^3}=\int \frac{x}{x+1}dx+C \quad \Rightarrow \quad \frac{y}{x^3}=\int [1+\frac{-1}{x+1}]dx+C,$$

$$\Rightarrow \quad \frac{y}{x^3}=x-\ln(x+1)+C \quad \text{or} \quad y=x^3[x-\ln(x+1)+C]. \qquad \text{(G.S)}$$

Example 2: Solve $(\sin^2 x-y)dx-\tan x dy=0$.

Solution :

$$(\sin^2 x-y)dx-\tan x dy=0 \quad \Rightarrow \quad \tan x\frac{dy}{dx}=\sin^2 x-y,$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sin^2 x}{\tan x}-\frac{y}{\tan x} \quad \Rightarrow \quad \frac{dy}{dx}+\frac{1}{\tan x}y=\frac{\sin^2 x}{\tan x}, \quad \text{(Linear DE with respect to } y\text{)}$$

where, $P(x)=\frac{1}{\tan x}=\frac{\cos x}{\sin x}$ and $Q(x)=\frac{\sin^2 x}{\tan x}=\sin x\cos x$.

$$\mu=e^{\int P(x)dx} \quad \Rightarrow \quad \mu=e^{\int \frac{\cos x}{\sin x}dx}=e^{\ln\sin x}=\sin x.$$

$$\mu.y=\int \mu.Qdx+C \quad \Rightarrow \quad \sin x.y=\int \sin x.(\sin x\cos x)dx+C_1,$$

$\Rightarrow \quad y\sin x = \int \sin^2 x \cos x dx + C_1 \quad \Rightarrow \quad y\sin x = \dfrac{\sin^3 x}{3} + C_1,$

or $\quad 3y\sin x = \sin^3 x + C. \quad [C = 3C_1] \quad$ (G.S)

Example 3: Solve $\quad (x-2y)dy + ydx = 0.$

Solution :

$(x-2y)dy + ydx = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} + \dfrac{y}{x-2y} = 0.$ (Nonlinear with respect to y)

But $\quad \dfrac{dx}{dy} + \dfrac{x-2y}{y} = 0 \quad \Rightarrow \quad \dfrac{dx}{dy} + \dfrac{1}{y}x = 2,$ (Linear DE with respect to x)

where, $\quad P(y) = \dfrac{1}{y} \quad$ and $\quad Q(y) = 2.$

$\mu = e^{\int P(y)dy} \quad \Rightarrow \quad \mu = e^{\int \frac{1}{y}dy} = e^{\ln y} = y.$

$\mu.x = \int \mu.Qdy + C \quad \Rightarrow \quad y.x = \int y.(2)dy + C,$

$\Rightarrow \quad xy = y^2 + C. \quad$ (G.S)