Reducible to exact differential equations

The differential equation $M(x,y)dx + N(x,y)dy = 0$ which is not exact (i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$) can be reduced to exact DE by multiplying it by a suitable function $\mu(x,y)$ which is called *integrating factor (I.F)*,

$$\mu M dx + \mu N dy = 0.$$

The above new DE is exact if $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$,

$$\therefore \mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}. \qquad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

The integrating factor $\mu(x,y)$ may be a function of *x* only, function of *y* only , or a function of both *x* and *y*.

There are two methods to find the integrating factor:

I) By equations

i- If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x only ($\mu(x,y)$ is a function of x only),

$\therefore\ \frac{\partial \mu}{\partial y}=0$ and $\frac{\partial \mu}{\partial x}=\frac{d\mu}{dx}$, then Eq.(1) becomes

$$\mu\frac{\partial M}{\partial y}=\mu\frac{\partial N}{\partial x}+N\frac{d\mu}{dx} \qquad \Rightarrow \qquad \mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=N\frac{d\mu}{dx},$$

$$\Rightarrow \qquad \frac{d\mu}{\mu}=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx \qquad \Rightarrow \qquad \ln\mu=\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx$$

$$\therefore \qquad \mu=e^{\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx}.$$

Example 1: Solve $(x+3y^2)dx+2xydy=0$.

Solution :

$$M(x,y)=x+3y^2 \qquad \Rightarrow \qquad \frac{\partial M}{\partial y}=6y,$$

$$N(x,y)=2xy \qquad \Rightarrow \qquad \frac{\partial N}{\partial x}=2y.$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the given DE is not exact.

Check, $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2xy}(6y-2y)=\frac{2}{x}$ (function of x only)

$$\therefore \mu=e^{\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx} \qquad \Rightarrow \qquad \mu=e^{\int\frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2.$$

Multiplying the given DE by the above integrating factor (I.F) gives

$$x^2(x+3y^2)dx+x^2(2xy)dy=0 \qquad \Rightarrow \qquad (x^3+3x^2y^2)dx+2x^3ydy=0.$$

Check, $M(x,y)=x^3+3x^2y^2 \qquad \Rightarrow \qquad \frac{\partial M}{\partial y}=6x^2y,$

$$N(x,y)=2x^3y \qquad \Rightarrow \qquad \frac{\partial N}{\partial x}=6x^2y.$$

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, then the given DE is reduced to exact one.

$$M=\frac{\partial f}{\partial x}=x^3+3x^2y^2 \quad \Rightarrow \quad f=\frac{x^4}{4}+x^3y^2+g(y),$$

$$\frac{\partial f}{\partial y}=2x^3y+g'(y), \qquad \text{but} \qquad \frac{\partial f}{\partial y}=N,$$

$$\therefore 2x^3y+g'(y)=N=2x^3y \quad \Rightarrow \quad g'(y)=0 \quad \Rightarrow \quad g(y)=C_1,$$

$$\therefore f=\frac{x^4}{4}+x^3y^2+C_1, \qquad \text{but} \qquad f=C_2,$$

$$\therefore \frac{x^4}{4}+x^3y^2+C_1=C_2 \quad \Rightarrow \quad \frac{x^4}{4}+x^3y^2=C_3, \qquad [C_3=C_2-C_1]$$

or $\quad x^4+4x^3y^2=C. \qquad [C=4C_3] \qquad$ (G.S)

Example 2: Solve $(\sin y+x^2+2x)dx=\cos y dy$.

Solution :

$$(\sin y+x^2+2x)dx=\cos y dy \quad \Rightarrow \quad (\sin y+x^2+2x)dx-\cos y dy=0.$$

$$M(x,y)=\sin y+x^2+2x \quad \Rightarrow \quad \frac{\partial M}{\partial y}=\cos y,$$

$$N(x,y)=-\cos y \quad \Rightarrow \quad \frac{\partial N}{\partial x}=0.$$

Since $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, then the given DE is not exact.

Check, $\quad \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=\dfrac{1}{-\cos y}\left(\cos y-0\right)=-1 \qquad$ (function of x only)

$$\therefore \mu=e^{\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx} \quad \Rightarrow \quad \mu=e^{\int(-1)dx}=e^{-x}.$$

Multiplying the given DE by the above integrating factor (I.F) gives

$$e^{-x}(\sin y+x^2+2x)dx-e^{-x}\cos y dy=0.$$

$$N=\frac{\partial f}{\partial y}=-e^{-x}\cos y \quad \Rightarrow \quad f=-e^{-x}\sin y+g(x),$$

$$\frac{\partial f}{\partial x}=e^{-x}\sin y+g'(x), \qquad \text{but} \qquad \frac{\partial f}{\partial x}=M,$$

$$\therefore e^{-x}\sin y+g'(x)=M=e^{-x}(\sin y+x^2+2x) \quad \Rightarrow \quad g'(x)=x^2e^{-x}+2xe^{-x},$$

$$\therefore g(x)=-x^2e^{-x}-2xe^{-x}-2e^{-x}-2xe^{-x}-2e^{-x} \Rightarrow g(x)=-x^2e^{-x}-4xe^{-x}-4e^{-x},$$

$$\therefore f=-e^{-x}\sin y-x^2e^{-x}-4xe^{-x}-4e^{-x}, \qquad \text{but} \qquad f=C,$$

$$\therefore -e^{-x}\sin y-x^2e^{-x}-4xe^{-x}-4e^{-x}=C_1,$$

or $\quad x^2+4x+4+\sin y=Ce^{x}. \qquad [C=-C_1] \qquad$ (G.S)

Note;

u		dv
x^2	+	e^{-x}
$2x$	-	$-e^{-x}$
2	+	e^{-x}
0		$-e^{-x}$

$=x^2(-e^{-x})-2xe^{-x}+2(-e^{-x})$

$=-x^2e^{-x}-2xe^{-x}-2e^{-x}$

u		dv
$2x$	+	e^{-x}
2	-	$-e^{-x}$
0		e^{-x}

$=2x(-e^{-x})-2e^{-x}$

$=-2xe^{-x}-2e^{-x}$

ii- If $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of y only ($\mu(x,y)$ is a function of y only),

$\therefore \quad \frac{\partial \mu}{\partial x}=0$ and $\frac{\partial \mu}{\partial y}=\frac{d\mu}{dy}$, then Eq.(1) becomes

$$\mu\frac{\partial M}{\partial y}+M\frac{d\mu}{dy}=\mu\frac{\partial N}{\partial x} \qquad \Rightarrow \qquad \mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-M\frac{d\mu}{dy},$$

$$\Rightarrow \quad \frac{d\mu}{\mu}=\frac{-1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dy \qquad \Rightarrow \qquad \ln\mu=\int\frac{-1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dy$$

$$\therefore \quad \mu=e^{-\int\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dy}.$$

Example 1: Solve $(y+2x)dx + x(y+x+1)dy = 0$.

Solution :

$$M(x,y) = y+2x \qquad \Rightarrow \qquad \frac{\partial M}{\partial y} = 1,$$

$$N(x,y) = xy + x^2 + x \qquad \Rightarrow \qquad \frac{\partial N}{\partial x} = y+2x+1.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the given DE is not exact.

Check, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1-(y+2x+1)}{xy+x^2+x} = \frac{-(y+2x)}{x(y+x+1)}$ (is not function of x only)

Check, $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1-(y+2x+1)}{y+2x} = \frac{-(y+2x)}{y+2x} = -1$ (function of y only)

$$\therefore \mu = e^{-\int \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dy} \qquad \Rightarrow \qquad \mu = e^{-\int(-1)dy} = e^{y}.$$

Multiplying the given DE by the above integrating factor (I.F) gives

$$e^{y}(y+2x)dx + xe^{y}(y+x+1)dy = 0.$$

$$M = \frac{\partial f}{\partial x} = ye^{y} + 2xe^{y} \qquad \Rightarrow \qquad f = xye^{y} + x^2e^{y} + g(y),$$

$$\frac{\partial f}{\partial y} = xye^{y} + xe^{y} + x^2e^{y} + g'(y), \qquad \text{but} \qquad \frac{\partial f}{\partial y} = N,$$

$$\therefore xye^{y} + xe^{y} + x^2e^{y} + g'(y) = N = xye^{y} + x^2e^{y} + xe^{y} \;\Rightarrow\; g'(y) = 0 \;\Rightarrow\; g(y) = C_1,$$

$$\therefore f = xye^{y} + x^2e^{y} + C_1, \qquad \text{but} \qquad f = C_2,$$

$$\therefore xye^{y} + x^2e^{y} + C_1 = C_2 \qquad \Rightarrow \qquad xy + x^2 = Ce^{-y}. \qquad [C = C_2 - C_1] \qquad \text{(G.S)}$$

Example 2: Solve $\cos y.dx + (2x\sin y - \cos^3 y).dy = 0$.

Solution :

$$M = \cos y \qquad \Rightarrow \qquad \frac{\partial M}{\partial y} = -\sin y,$$

$$N = 2x\sin y - \cos^3 y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2\sin y .$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the given DE is not exact.

Check, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-\sin y - 2\sin y}{2x\sin y - \cos^3 y} = \frac{-3\sin y}{2x\sin y - \cos^3 y}$ (is not function of *x* only)

Check, $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-\sin y - 2\sin y}{\cos y} = \frac{-3\sin y}{\cos y}$, (function of *y* only)

$$\therefore \mu = e^{-\int \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dy} \quad \Rightarrow \quad \mu = e^{-\int \frac{-3\sin y}{\cos y}dy} = e^{-3\ln\cos y} = \cos^{-3} y = \frac{1}{\cos^3 y} .$$

Multiplying the given DE by the above integrating factor (I.F) gives

$$\frac{1}{\cos^3 y}(\cos y).dx + \frac{1}{\cos^3 y}(2x\sin y - \cos^3 y).dy = 0 \Rightarrow \frac{1}{\cos^2 y}.dx + \left(\frac{2x\sin y}{\cos^3 y} - 1\right).dy = 0$$

$$M = \frac{\partial f}{\partial x} = \frac{1}{\cos^2 y} \quad \Rightarrow \quad f = \frac{x}{\cos^2 y} + g(y) ,$$

$$\frac{\partial f}{\partial y} = -2x\cos^{-3}.(-\sin y) + g'(y) = \frac{2x\sin y}{\cos^3 y} + g'(y) , \quad \text{but} \quad \frac{\partial f}{\partial y} = N ,$$

$$\therefore \frac{2x\sin y}{\cos^3 y} + g'(y) = N = \frac{2x\sin y}{\cos^3 y} - 1 \quad \Rightarrow \quad g'(y) = -1 \quad \Rightarrow \quad g(y) = -y ,$$

$$\therefore f = \frac{x}{\cos^2 y} - y , \quad \text{but} \quad f = C ,$$

$$\therefore \frac{x}{\cos^2 y} - y = C , \quad \text{or} \quad x = (y + C)\cos^2 y . \qquad \text{(G.S)}$$

iii- If $\mu(x, y)$ is a function of *x* and *y*, then a partial DE should be used.