Reducible to homogeneous DE

Consider the DE $(a_1x+b_1y+c_1)dx \pm (a_2x+b_2y+c_2)dy=0.$

If $c_1=c_2=0$, then the given DE is homogeneous.

If $c_1 \neq 0$ or $c_2 \neq 0$, the given DE is nonhomogeneous, then consider the lines:

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0.$$

***** If $\left(\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\right)$, then the two lines intersect at a point such as $p(h,k)$, and the given DE can be reduced to a homogeneous DE by the two substitutions:

$$x=x^*+h \quad \text{and} \quad y=y^*+k.$$

***** If $\left(\frac{a_1}{a_2} = \frac{b_1}{b_2} = r\right)$, then the two lines are parallel, and the given DE becomes

$$[r(a_2x+b_2y)+c_1]dx \pm [(a_2x+b_2y)+c_2]dy=0,$$

and the given DE can be reduced to a separable variables DE by the substitution:

$$z=a_2x+b_2y.$$

Example 1: Solve $(x-4y-3)dx-(x-6y-5)dy=0.$

Solution :

$$\frac{a_1}{a_2}=\frac{1}{1}=1 \quad \text{and} \quad \frac{b_1}{b_2}=\frac{-4}{-6}=\frac{2}{3}.$$

Since $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the two lines intersect. To find the point of intersection,

$$x-4y-3=0, \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$x-6y-5=0. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

Subtracting Eq.2 from Eq.1 gives $2y+2=0 \Rightarrow y=-1 \Rightarrow x=-1.$

Thus the point of intersection $p(h,k)$ is $p(-1,-1)$.

$\therefore$ Let $x=x^*+h=x^*-1 \quad \Rightarrow \quad dx=dx^*,$

and $y=y^*+k=y^*-1 \quad \Rightarrow \quad dy=dy^*,$

$\therefore \quad [(x^*-1)-4(y^*-1)-3]dx^* - [(x^*-1)-6(y^*-1)-5]dy^* = 0,$

$\Rightarrow \quad (x^*-4y^*)dx^* - (x^*-6y^*)dy^* = 0.$ (Homogeneous DE)

Let $y^* = ux^* \quad \Rightarrow \quad dy^* = u.dx^* + x^*.du,$

$\therefore \quad [x^*-4(ux^*)]dx^* - [x^*-6(ux^*)](u.dx^* + x^*.du) = 0,$

$\Rightarrow \quad (1-5u+6u^2)dx^* - x^*(1-6u)du = 0,$ (Separable DE)

$$\frac{dx^*}{x^*} - \frac{1-6u}{6u^2-5u+1}du = 0 \quad \Rightarrow \quad \int \frac{dx^*}{x^*} - \int \frac{1-6u}{6u^2-5u+1}du = \int 0.$$

For $\dfrac{1-6u}{6u^2-5u+1} = \dfrac{1-6u}{(3u-1)(2u-1)} = \dfrac{A}{(3u-1)} + \dfrac{B}{(2u-1)},$

$\Rightarrow \quad 1-6u = A(2u-1) + B(3u-1),$

At $u = \frac{1}{3} \quad \Rightarrow \quad 1-6\left(\frac{1}{3}\right) = A\left[2\left(\frac{1}{3}\right)-1\right]+0 \quad \Rightarrow \quad -1 = A\left(\frac{-1}{3}\right) \quad \Rightarrow \quad A = 3.$

At $u = \frac{1}{2} \quad \Rightarrow \quad 1-6\left(\frac{1}{2}\right) = 0 + B\left[3\left(\frac{1}{2}\right)-1\right] \quad \Rightarrow \quad -2 = B\left(\frac{1}{2}\right) \quad \Rightarrow \quad B = -4.$

$$\therefore \int \frac{dx^*}{x^*} - \int\left[\frac{3}{(3u-1)} + \frac{-4}{(2u-1)}\right]du = \int 0 \quad \Rightarrow \quad \ln x^* - \ln(3u-1) + 2\ln(2u-1) = C_1,$$

$$\Rightarrow \ln\left[\frac{x^*(2u-1)^2}{(3u-1)}\right] = C_1 \quad \Rightarrow \quad \frac{x^*(2u-1)^2}{(3u-1)} = e^{C_1} \quad \Rightarrow \quad \frac{x^*\left[2\left(\frac{y^*}{x^*}\right)-1\right]^2}{\left[3\left(\frac{y^*}{x^*}\right)-1\right]} = C \quad [C = e^{C_1}],$$

$$\Rightarrow \quad \frac{x^*\left[\frac{2y^*-x^*}{x^*}\right]^2}{\left[\frac{3y^*-x^*}{x^*}\right]} = C \quad \Rightarrow \quad \frac{(2y^*-x^*)^2}{(3y^*-x^*)} = C,$$

$$\Rightarrow \quad \frac{[2(y+1)-(x+1)]^2}{[3(y+1)-(x+1)]} = C \quad \Rightarrow \quad (2y-x+1)^2 = C(3y-x+2). \qquad \text{(G.S)}$$

Example 2: Solve $(2x+3y+4)dx - (4x+6y+1)dy = 0.$

Solution :

$$\frac{a_1}{a_2}=\frac{2}{4}=\frac{1}{2} \quad \text{and} \quad \frac{b_1}{b_2}=\frac{3}{6}=\frac{1}{2}.$$

Since $\frac{a_1}{a_2}=\frac{b_1}{b_2}$, then the two lines are parallel.

$$(2x+3y+4)dx-[2(2x+3y)+1]dy=0,$$

Let $z=2x+3y \Rightarrow dz=2dx+3dy \Rightarrow dy=\frac{1}{3}(dz-2dx)$,

$$\therefore \quad (z+4)dx-(2z+1)[\frac{1}{3}(dz-2dx)]=0,$$

$$\Rightarrow \quad 7(z+2)dx-(2z+1)dz=0, \qquad \text{(Separable DE)}$$

$$\Rightarrow \quad 7dx-\frac{2z+1}{z+2}dz=0 \quad \Rightarrow \quad \int 7dx-\int\frac{2z+1}{z+2}dz=\int 0.$$

For $\frac{2z+1}{z+2}=\frac{2(z+2)-3}{z+2}=2-\frac{3}{z+2}$,

$$\therefore \quad \int 7dx-\int\left[2-\frac{3}{z+2}\right]dz=\int 0 \quad \Rightarrow \quad 7x-2z+3\ln(z+2)=C_1,$$

$$\Rightarrow \quad 7x-2(2x+3y)+3\ln(2x+3y+2)=C_1 \qquad \text{or}$$

$$\ln(2x+3y+2)=2y-x+\frac{C_1}{3},$$

$$\Rightarrow \quad \ln(2x+3y+2)=2y-x+C. \qquad [C=\frac{C_1}{3}] \qquad \text{(G.S)}$$

3- Exact differential equations

Theorem: The differential equation $M(x,y)dx+N(x,y)dy=0$ is exact if and only if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

Proof: Let $f(x,y)=C$ be any function, then the total differentiation (exact differential) of $f(x,y)$ is given by;

$$df=\frac{\partial f}{\partial x}.dx+\frac{\partial f}{\partial y}.dy=0.$$

Let $\frac{\partial f}{\partial x}=M(x,y)$ and $\frac{\partial f}{\partial y}=N(x,y)$, then

$$\therefore M(x,y).dx+N(x,y).dy=0.$$

We have $\frac{\partial M}{\partial y}=\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial N}{\partial x}=\frac{\partial^2 f}{\partial x \partial y}$.

But $\frac{\partial^2 f}{\partial y \partial x}=\frac{\partial^2 f}{\partial x \partial y}$ $\Rightarrow$ $\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

To solve an exact DE we use the following procedure;

$$\frac{\partial f}{\partial x}=M(x,y) \quad \Rightarrow \quad f(x,y)=\int M(x,y).dx+g(y),$$

Since $f(x,y)=C$ $\Rightarrow$ $\int M(x,y).dx+g(y)=C$.

To find $g(y)$,

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left[\int M(x,y).dx\right]+g'(y), \quad \text{but} \quad \frac{\partial f}{\partial y}=N(x,y),$$

$$\therefore \frac{\partial}{\partial y}\left[\int M(x,y).dx\right]+g'(y)=N(x,y) \quad \Rightarrow \quad g'(y)=N(x,y)-\frac{\partial}{\partial y}\left[\int M(x,y).dx\right],$$

$$\therefore g(y)=\int\left[N(x,y)-\frac{\partial}{\partial y}\left[\int M(x,y).dx\right]\right]dy.$$

Example 1: Solve $(3x^2y+2xy)dx+(x^3+x^2+2y)dy=0$.

Solution :

$$M(x,y)=3x^2y+2xy \quad \Rightarrow \quad \frac{\partial M}{\partial y}=3x^2+2x,$$

$$N(x,y)=x^3+x^2+2y \quad \Rightarrow \quad \frac{\partial N}{\partial x}=3x^2+2x.$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the given DE is an exact DE.

*** *Solution I;***

$$M(x,y)=\frac{\partial f}{\partial x}=3x^2y+2xy \quad \Rightarrow \quad f(x,y)=x^3y+x^2y+g(y),$$

$$\frac{\partial f}{\partial y}=x^3+x^2+g'(y), \quad \text{but} \quad \frac{\partial f}{\partial y}=N(x,y),$$

$\therefore x^3 + x^2 + g'(y) = N(x,y) = x^3 + x^2 + 2y \quad \Rightarrow \quad g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2,$

$\therefore f(x,y) = x^3y + x^2y + y^2,$ but $f(x,y) = C,$

$\therefore x^3y + x^2y + y^2 = C.$ (G.S)

*** Solution II;**

$$N(x,y) = \frac{\partial f}{\partial y} = x^3 + x^2 + 2y \quad \Rightarrow \quad f(x,y) = x^3y + x^2y + y^2 + q(x),$$

$$\frac{\partial f}{\partial x} = 3x^2y + 2xy + q'(x), \quad \text{but} \quad \frac{\partial f}{\partial x} = M(x,y),$$

$$\therefore 3x^2y + 2xy + q'(x) = M(x,y) = 3x^2y + 2xy \quad \Rightarrow \quad q'(x) = 0 \quad \Rightarrow \quad q(x) = C_1,$$

$$\therefore f(x,y) = x^3y + x^2y + y^2 + C_1, \quad \text{but} \quad f(x,y) = C_2,$$

$$\therefore x^3y + x^2y + y^2 + C_1 = C_2 \quad \Rightarrow \quad x^3y + x^2y + y^2 = C. \quad [C = C_2 - C_1] \quad \text{(G.S)}$$

*** Solution III;**

$$M(x,y) = \frac{\partial f}{\partial x} = 3x^2y + 2xy \quad \Rightarrow \quad f(x,y) = x^3y + x^2y + g(y),$$

$$N(x,y) = \frac{\partial f}{\partial y} = x^3 + x^2 + 2y \quad \Rightarrow \quad f(x,y) = x^3y + x^2y + y^2 + q(x),$$

Comparing the above two expressions of $f(x,y)$ yields,

$g(y) = y^2$ and $q(x) = 0,$

$\therefore f(x,y) = x^3y + x^2y + y^2,$ but $f(x,y) = C,$

$\therefore x^3y + x^2y + y^2 = C.$ (G.S)

Example 2: Solve $(x^2 \cos xy + e^y)dy + (xy\cos xy + \sin xy)dx = 0.$

Solution :

$$M(x,y) = xy\cos xy + \sin xy \quad \Rightarrow \quad \frac{\partial M}{\partial y} = x[y(-\sin xy)x + \cos xy] + \cos xy(x),$$

$$= -x^2 y \sin xy + 2x \cos xy ,$$

$$N(x,y) = x^2 \cos xy + e^y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = x^2(-\sin xy)y + \cos xy(2x) + 0 ,$$

$$= -x^2 y \sin xy + 2x \cos xy .$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then the given DE is an exact DE.

$$N = \frac{\partial f}{\partial y} = x^2 \cos xy + e^y \quad \Rightarrow \quad f = x \sin xy + e^y + g(x) ,$$

$$\frac{\partial f}{\partial x} = x(\cos xy)y + \sin xy + 0 + g'(x) , \quad \text{but} \quad \frac{\partial f}{\partial x} = M ,$$

$$\therefore xy \cos xy + \sin xy + g'(x) = M = xy \cos xy + \sin xy \quad \Rightarrow \quad g'(x) = 0 \quad \Rightarrow \quad g(x) = C_1 ,$$

$$\therefore f = x \sin xy + e^y + C_1 , \quad \text{but} \quad f = C_2 ,$$

$$\therefore x \sin xy + e^y + C_1 = C_2 \quad \Rightarrow \quad x \sin xy + e^y = C . \quad [C = C_2 - C_1] \quad \text{(G.S)}$$