2- First-Order Ordinary Differential Equations

Introduction

A differential equation of the first order and first degree may be written in one of the following two forms

1- $\dfrac{dy}{dx} = f(x, y)$ (The slope form)

2- $M(x, y)dx + N(x, y)dy = 0$ (The exact form)

For example, the following DE

$$\frac{dy}{dx} = \frac{2x - y}{y - 3x} \quad \text{may be rewritten as} \quad (2x - y)dx - (y - 3x)dy = 0,$$

where, $f(x, y) = \dfrac{2x - y}{y - 3x}$, $M(x, y) = 2x - y$, and $N(x, y) = -(y - 3x)$.

1- Separable variables differential equations

If we can separate the dependent variable and its differential from the independent variable and its differential in a DE, then this DE is called "separable variables DE". For example, if a given first-order DE can be reduced to

$$\frac{dy}{dx} = f(x, y) \quad \Rightarrow \quad \frac{dy}{dx} = g(x).h(y) \quad \Rightarrow \quad \frac{1}{h(y)}.dy = g(x).dx.$$

Or $M(x, y)dx + N(x, y)dy = 0 \quad \Rightarrow \quad g_1(x).h_1(y).dx + g_2(x).h_2(y).dy = 0$,

$$\Rightarrow \quad \frac{g_1(x)}{g_2(x)}.dx + \frac{h_2(y)}{h_1(y)}.dy = 0,$$

then such equation is called "separable variables DE" which can be solved by integrating both sides.

Example 1: Solve the following first-order differential equation

$$\frac{dy}{dx} = \frac{-x}{y}.$$

Solution :

$\frac{dy}{dx} = \frac{-x}{y}$ (separable variables DE) $\Rightarrow$ $y.dy = -x.dx$,

$\therefore \int y.dy = \int -x dx \Rightarrow \frac{y^2}{2} = \frac{-x^2}{2} + C_1$,

or $y^2 = -x^2 + 2C_1 \Rightarrow y^2 + x^2 = 2C_1$

$\therefore y^2 + x^2 = C.$ $[C = 2C_1]$ (General solution G.S)

Notes:

***** The previous DE may be given in different forms, like

$y' = \frac{-x}{y}$ or $Dy = \frac{-x}{y}$ or $y.dy + x.dx = 0$.

***** As a check, we try to find *dy/dx* by differentiating the general solution,

$y^2 + x^2 = C \Rightarrow 2y.dy + 2xdx = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}.$ O.K

Example 2: Solve $(1 + x^3).dy - x^2 y.dx = 0$.

Solution:

$(1 + x^3).dy - x^2 y.dx = 0$ (separable variables DE) $\Rightarrow \frac{dy}{y} - \frac{x^2}{1 + x^3}.dx = 0$,

$\therefore \int \frac{dy}{y} - \int \frac{x^2}{1 + x^3}.dx = \int 0 \Rightarrow \ln y - \frac{1}{3}\ln(1 + x^3) = C_1$,

or $3\ln y - \ln(1 + x^3) = 3C_1 \Rightarrow \ln y^3 - \ln(1 + x^3) = C_2$, $[C_2 = 3C_1]$

$\Rightarrow \ln \frac{y^3}{1 + x^3} = C_2 \Rightarrow \frac{y^3}{1 + x^3} = e^{C_2} \Rightarrow \frac{y^3}{1 + x^3} = C$ $[C = e^{C_2}]$

$\therefore y^3 = C(1 + x^3).$ (G.S)

Example 3: Solve $y' = xy - x$, $y(0) = 3$.

Solution:

$$\frac{dy}{dx} = xy - x \quad \Rightarrow \quad \frac{dy}{dx} = x(y-1) \qquad \text{(separable variables DE)}$$

$$\therefore \quad \frac{dy}{y-1} = xdx \quad \Rightarrow \quad \int \frac{dy}{y-1} = \int xdx \quad \Rightarrow \quad \ln(y-1) = \frac{x^2}{2} + C_1,$$

or $y - 1 = e^{x^2/2 + C_1} \Rightarrow y - 1 = e^{x^2/2}.e^{C_1} \Rightarrow y - 1 = Ce^{x^2/2}$, $[C = e^{C_1}]$

$\therefore \quad y = 1 + Ce^{x^2/2}$. (G.S)

Apply the given condition, at $x = 0$, $y = 3 \Rightarrow 3 = 1 + Ce^{0^2/2} \Rightarrow C = 2$.

$\therefore \quad y = 1 + 2e^{x^2/2}$. (P.S)

Example 4: Solve $\dfrac{dy}{dx} = -2 + e^{2x+y-1}$.

Solution:

Let $z = 2x + y - 1 \Rightarrow dz = 2dx + dy \Rightarrow \dfrac{dy}{dx} = \dfrac{dz}{dx} - 2$,

$$\therefore \quad \frac{dz}{dx} - 2 = -2 + e^z \quad \Rightarrow \quad \frac{dz}{dx} = e^z, \qquad \text{(separable variables DE)}$$

$$\frac{dz}{e^z} = dx \quad \Rightarrow \quad \int \frac{dz}{e^z} = \int dx \quad \Rightarrow \quad -e^{-z} = x + C_1,$$

or $x + e^{-z} = -C_1 \Rightarrow x + e^{-2x-y+1} = C$. $[C = -C_1]$ (G.S)

Example 5: Solve $x(2xy+1)dy + y(1 + 2xy - x^3y^3)dx = 0$.

Solution:

Let $z = xy \Rightarrow dz = xdy + ydx \Rightarrow dy = \dfrac{dz - ydx}{x} \Rightarrow dy = \dfrac{dz - \frac{z}{x}dx}{x}$,

$$\therefore x(2z+1)\frac{dz - \frac{z}{x}dx}{x} + \frac{z}{x}(1 + 2z - z^3)dx = 0 \Rightarrow (2z+1).dz - \frac{z^4}{x}.dx = 0, \text{ (separable)}$$

$$\frac{2z+1}{z^4}.dz - \frac{dx}{x} = 0 \;\Rightarrow\; \int \frac{2z+1}{z^4}.dz - \int \frac{dx}{x} = \int 0 \;\Rightarrow\; \int\left(\frac{2}{z^3} + \frac{1}{z^4}\right).dz - \int \frac{dx}{x} = \int 0,$$

$$-\frac{1}{z^2} - \frac{1}{3z^3} - \ln x = C_1 \;\Rightarrow\; \frac{1}{(xy)^2} + \frac{1}{3(xy)^3} + \ln x = C. \quad [C = -C_1] \qquad \text{(G.S)}$$

2- Homogeneous differential equations (reducible to separable DE)

A function $f(x,y)$ is said to be homogeneous of degree n if;

$$f(tx,ty) = t^n f(x,y).$$

For example,

* If $f(x,y) = 2y^4 - x^2y^2$, then

$$f(tx,ty) = 2(ty)^4 - (tx)^2(ty)^2 = t^4(2y^4 - x^2y^2) = t^4 f(x,y),$$

$\therefore$ $f(x,y)$ is homogeneous of degree 4.

* If $f(x,y) = \frac{y}{x} - 3e^{x/y} + \sin\frac{x}{y}$, then

$$f(tx,ty) = \frac{ty}{tx} - 3e^{tx/ty} + \sin\frac{tx}{ty} = t^0\left(\frac{y}{x} - 3e^{x/y} + \sin\frac{x}{y}\right) = t^0 f(x,y),$$

$\therefore$ $f(x,y)$ is homogeneous of degree 0.

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is called homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous and of the same degree (i.e. all terms of the DE are of the same total degree in the variables *x* and *y*).

For example,

* $x(y+x)dx = y^2dy$, is homogeneous of degree 2.
* $(xy+x)dx = (y^2 - x^2)dy$, is non-homogeneous.
* $y(\ln y - \ln x - 1)dx + xdy = 0$, is homogeneous of degree 1.
* $y(\ln y - 1)dx + xdy = 0$, is non-homogeneous.

The homogeneous DE can be always reduced to separable variables DE by the substitution $y = ux$ or $x = uy$.

Example 1: Solve $2(2x^2+y^2)dx-xydy=0$.

Solution :

The given DE is homogeneous of degree 2.

Let $y=ux \quad \Rightarrow \quad dy=udx+xdu$,

$$\therefore 2(2x^2+(ux)^2)dx-x(ux)(udx+xdu)=0$$

$$\Rightarrow (4x^2+2u^2x^2)dx-u^2x^2dx-ux^3du=0,$$

$$\Rightarrow \quad (4x^2+u^2x^2)dx-ux^3du=0 \quad \Rightarrow \quad (4+u^2)dx-uxdu=0, \qquad \text{(Separable DE)}$$

$$\frac{dx}{x}-\frac{u}{4+u^2}du=0 \quad \Rightarrow \quad \ln x-\frac{1}{2}\ln(4+u^2)=C_1,$$

or $\quad 2\ln x-\ln(4+u^2)=2C_1 \quad \Rightarrow \quad \ln\frac{x^2}{4+u^2}=2C_1 \quad \Rightarrow \quad \frac{x^2}{4+u^2}=e^{2C_1}$,

$$\Rightarrow \frac{x^2}{4+u^2}=C \quad [C=e^{2C_1}] \quad \Rightarrow \quad x^2=C(4+u^2) \quad \Rightarrow \quad x^2=C(4+(\frac{y}{x})^2),$$

$$\therefore \quad x^4=C(4x^2+y^2). \qquad \text{(G.S)}$$

Note:

***** The given DE can also be solved by letting $x=uy$.

Example 2: Solve $ydx+\left[y\cos^2\left(\frac{x}{y}\right)-x\right]dy=0$.

Solution :

The given DE is homogeneous of degree 1.

Let $x=uy \quad \Rightarrow \quad dx=udy+ydu$,

$$\therefore y(udy+ydu)+\left[y\cos^2\left(\frac{uy}{y}\right)-uy\right]dy=0 \quad \Rightarrow \quad yudy+y^2du+y\cos^2udy-uydy=0,$$

$$\Rightarrow \quad y^2du+y\cos^2udy=0 \quad \Rightarrow \quad ydu+\cos^2udy=0, \qquad \text{(Separable DE)}$$

$$\frac{du}{\cos^2u}+\frac{dy}{y}=0 \quad \Rightarrow \quad \sec^2udu+\frac{dy}{y}=0 \quad \Rightarrow \quad \tan u+\ln y=C,$$

$$\therefore \quad \tan\left(\frac{x}{y}\right)+\ln y=C. \qquad \text{(G.S)}$$