Multiple Integrals

15.1

Double and Iterated Integrals over Rectangles

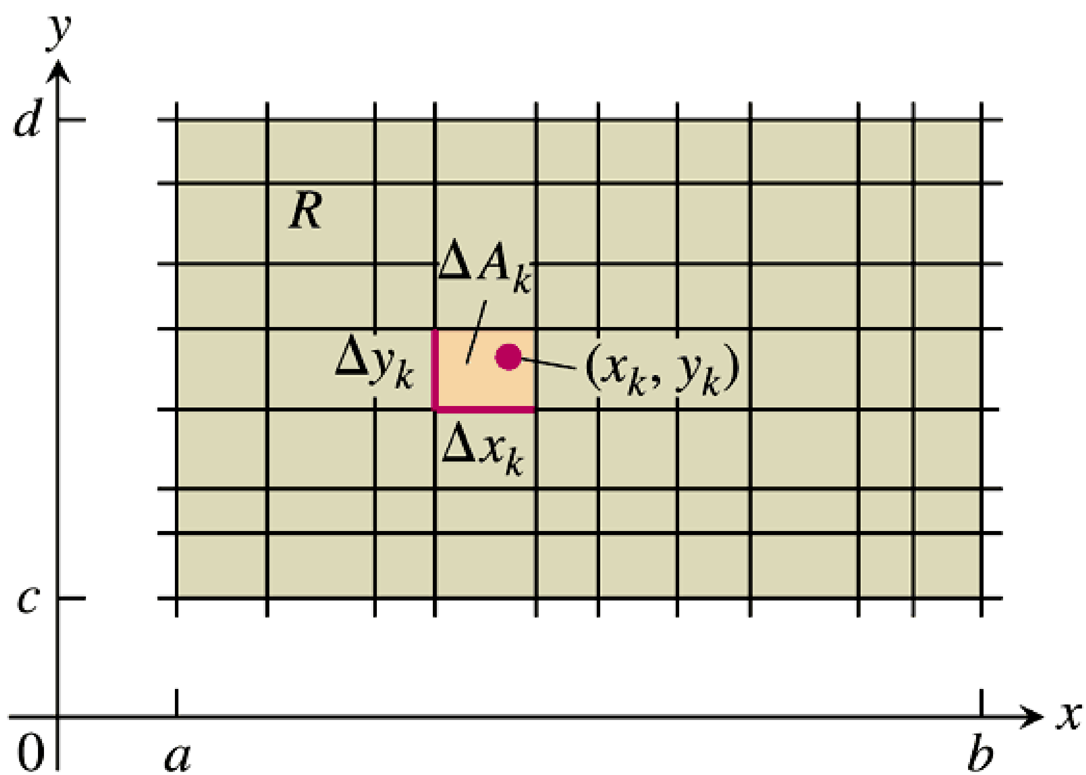


FIGURE 15.1 Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \, \Delta y_k$.

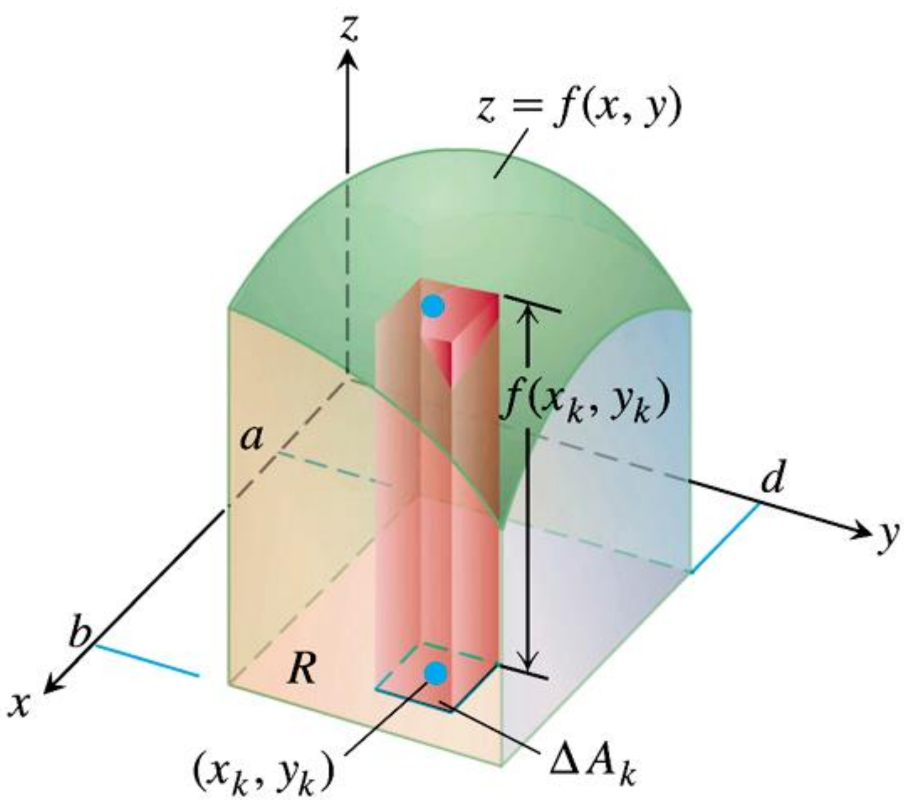


FIGURE 15.2 Approximating solids with rectangular boxes leads us to define the volumes of more general solids as double integrals. The volume of the solid shown here is the double integral of $f(x, y)$ over the base region R.

$$\text{Volume} = \lim_{n \to \infty} S_n = \iint_R f(x, y)\, dA,$$

where $\Delta A_k \to 0$ as $n \to \infty$.

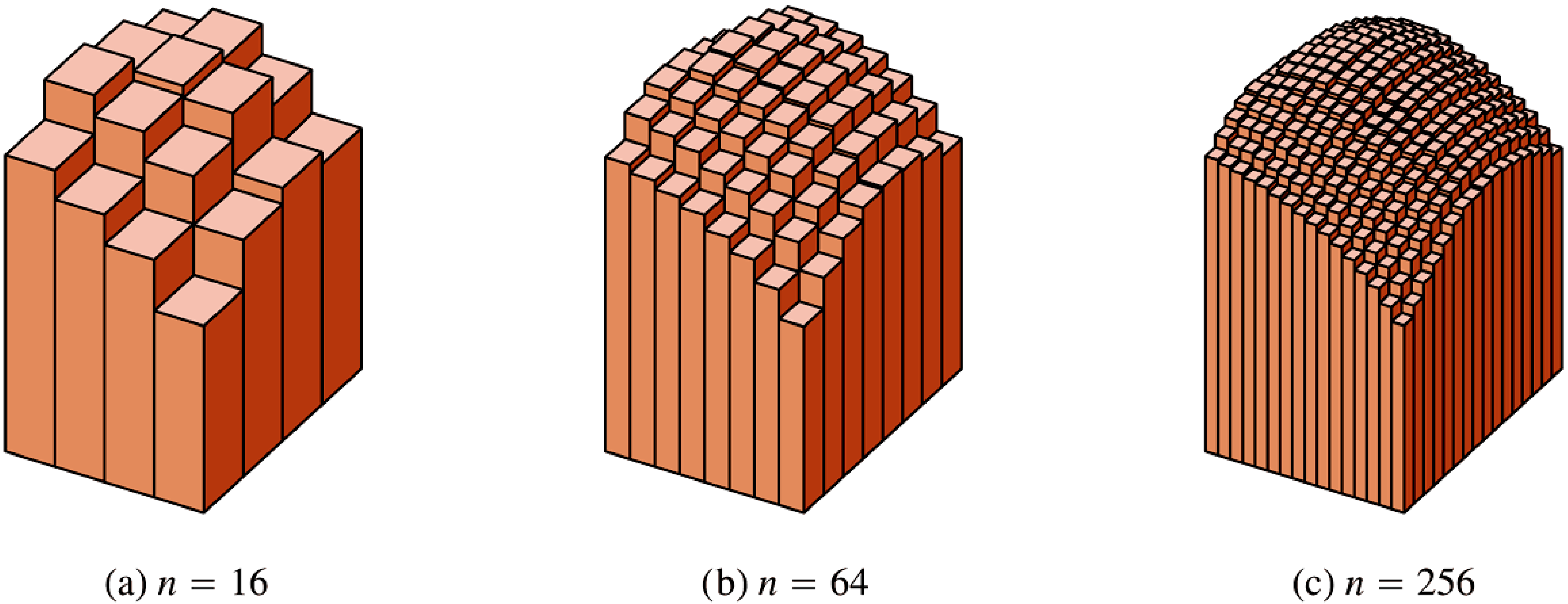


FIGURE 15.3 As n increases, the Riemann sum approximations approach the total volume of the solid shown in Figure 15.2.

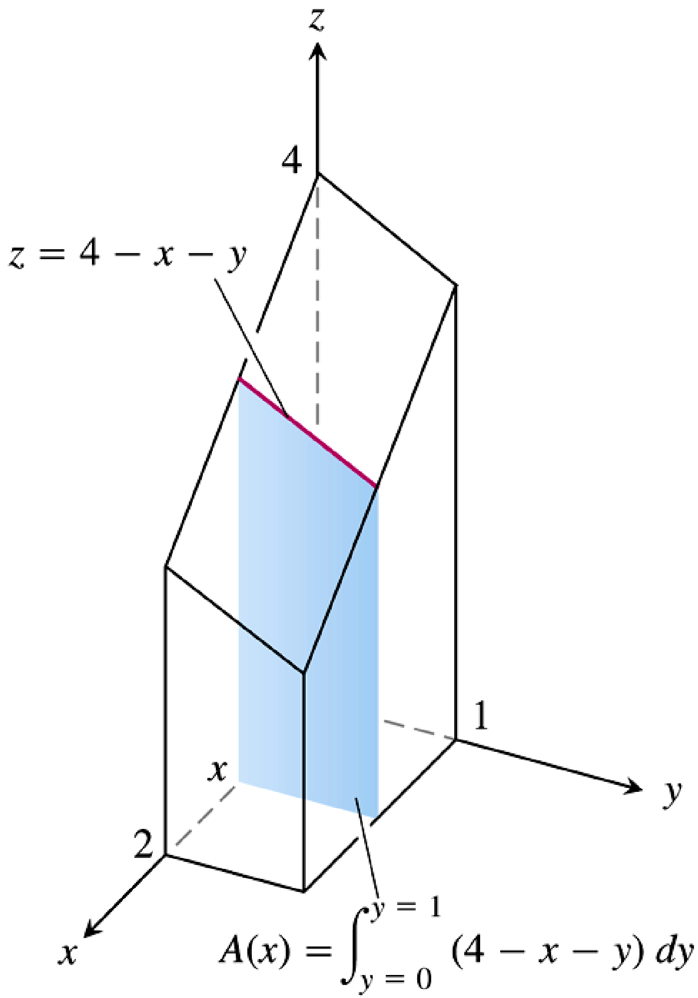


FIGURE 15.4 To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

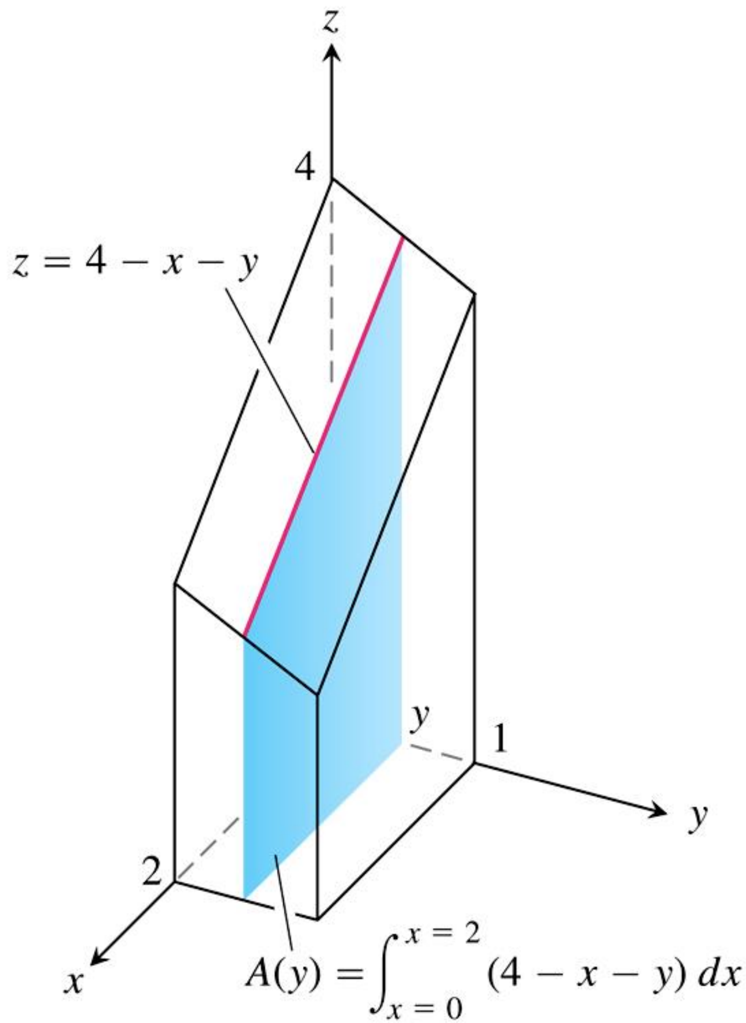


FIGURE 15.5 To obtain the cross-sectional area $A(y)$, we hold y fixed and integrate with respect to x.

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \qquad \text{and} \qquad R: \quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution Figure 15.6 displays the volume beneath the surface. By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_{0}^{2} (100 - 6x^2y)\, dx\, dy = \int_{-1}^{1} \Big[100x - 2x^3y\Big]_{x=0}^{x=2}\, dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy = \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$

Reversing the order of integration gives the same answer:

$$\int_0^2\int_{-1}^{1} (100 - 6x^2y)\, dy\, dx = \int_0^2 \Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1}\, dx$$

$$= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)]\, dx$$

$$= \int_0^2 200\, dx = 400.$$

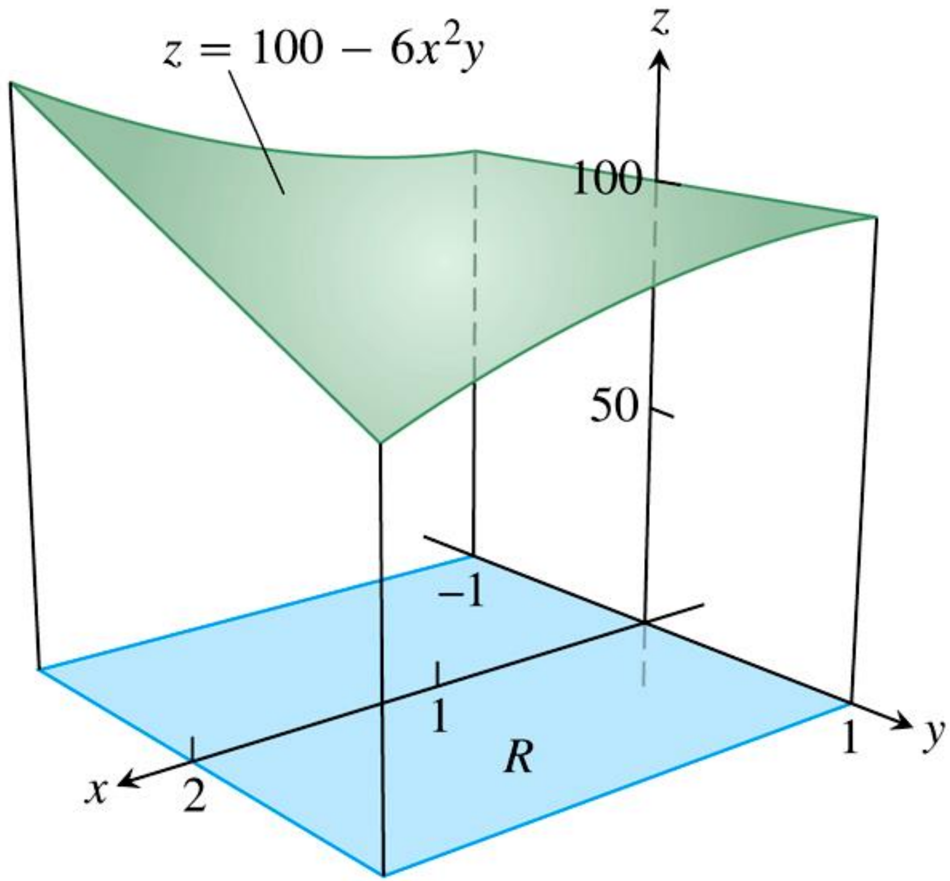


FIGURE 15.6 The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R (Example 1).

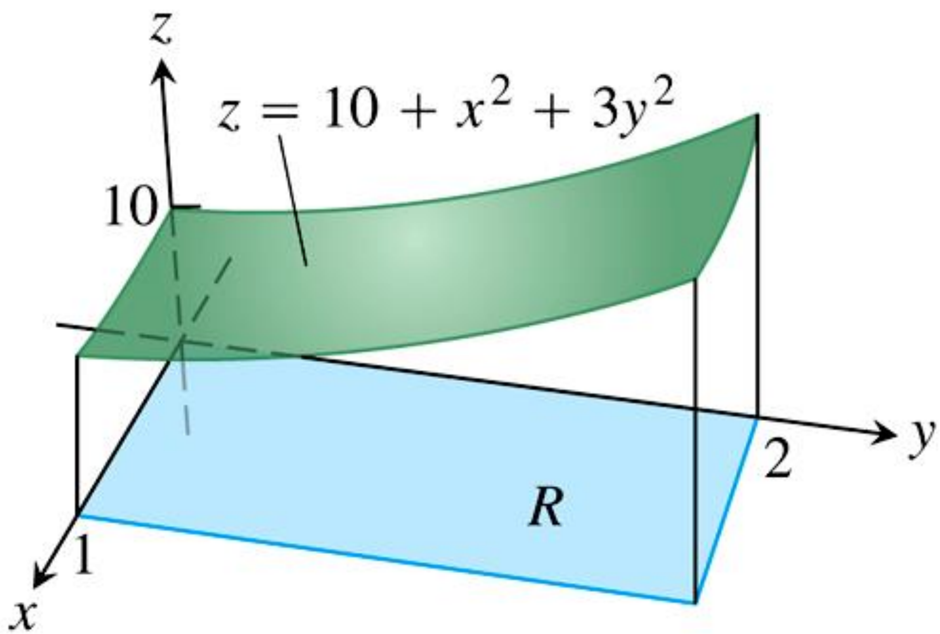


FIGURE 15.7 The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R (Example 2).

15.2

Double Integrals over General Regions

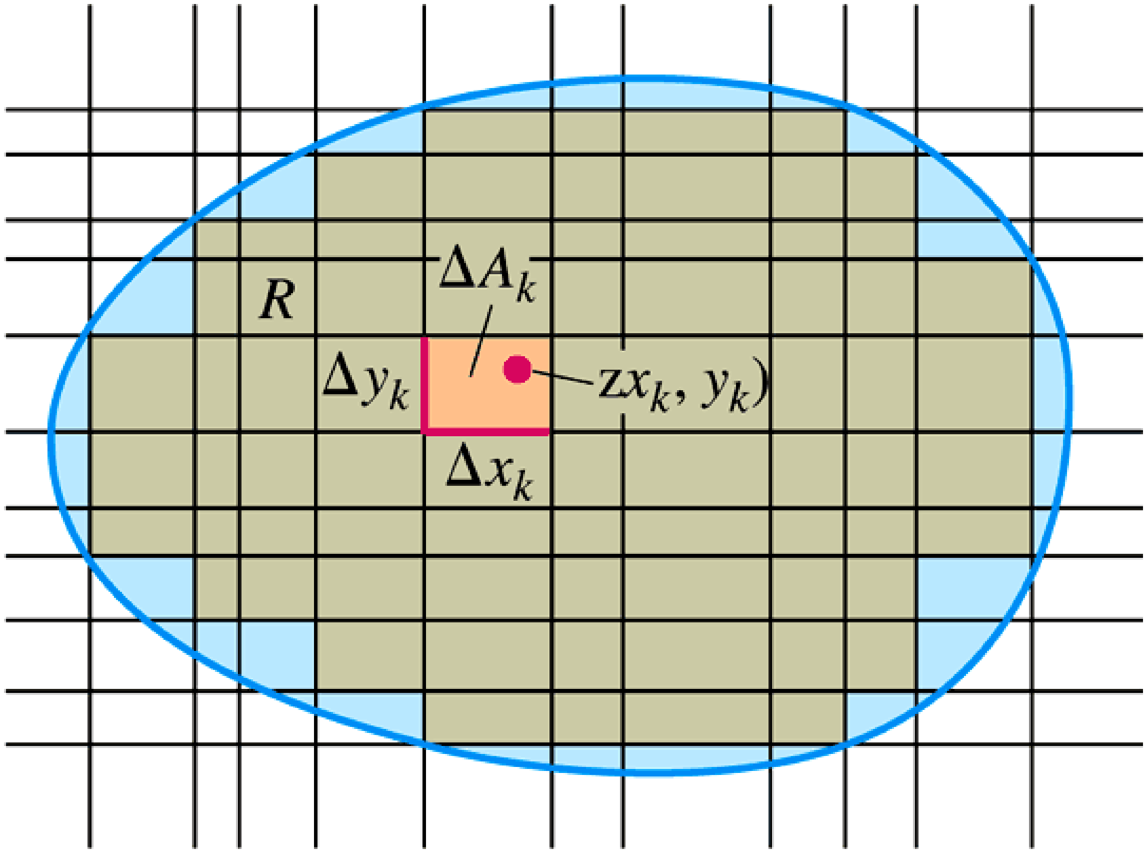


FIGURE 15.8 A rectangular grid partitioning a bounded nonrectangular region into rectangular cells.

If $f(x, y)$ is positive and continuous over R, we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)\, dA$, as before (Figure 15.9).

If R is a region like the one shown in the xy-plane in Figure 15.10, bounded "above" and "below" by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a$, $x = b$, we may again calculate the volume by the method of slicing. We first calculate the cross-sectional area

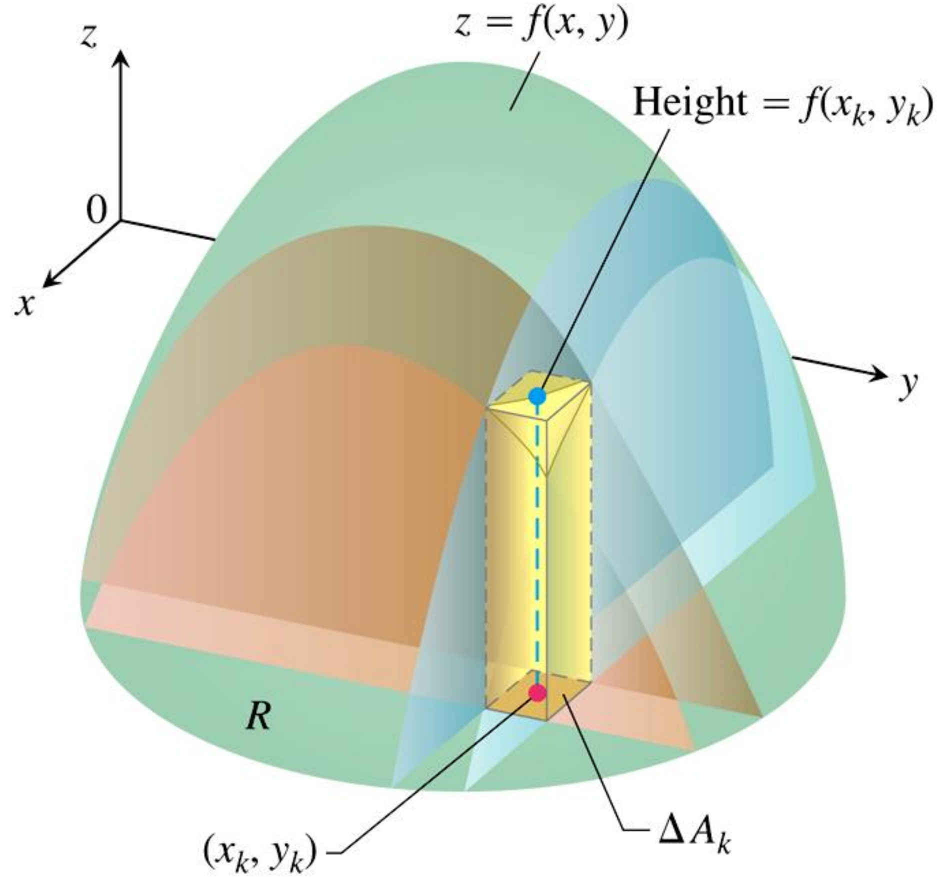
$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy$$

and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx. \tag{1}$$

Similarly, if R is a region like the one shown in Figure 15.11, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines $y = c$ and $y = d$, then the volume calculated by slicing is given by the iterated integral

$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy. \tag{2}$$



$$\text{Volume} = \lim \sum f(x_k, y_k)\,\Delta A_k = \iint_R f(x, y)\,dA$$

FIGURE 15.9 We define the volumes of solids with curved bases as a limit of approximating rectangular boxes.

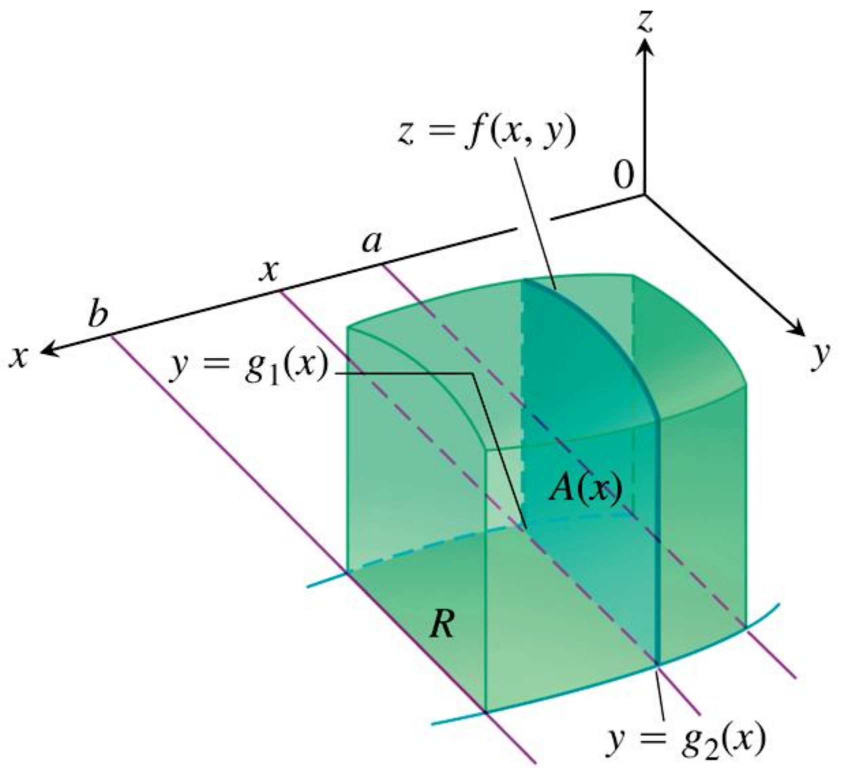


FIGURE 15.10 The area of the vertical slice shown here is $A(x)$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$:

$$\int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

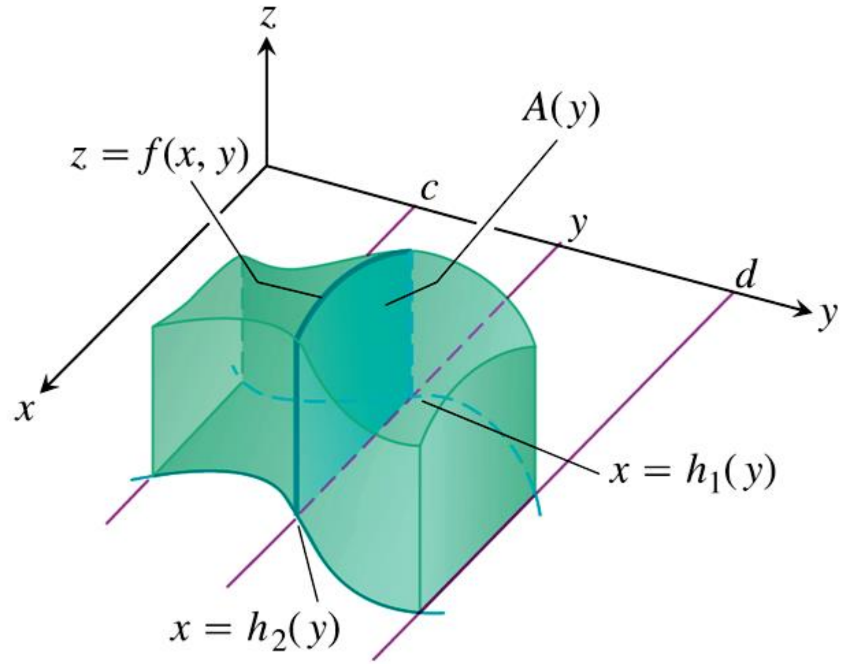


FIGURE 15.11 The volume of the solid shown here is

$$\int_c^d A(y)\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

For a given solid, Theorem 2 says we can calculate the volume as in Figure 15.10, or in the way shown here. Both calculations have the same result.

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure 15.12. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$

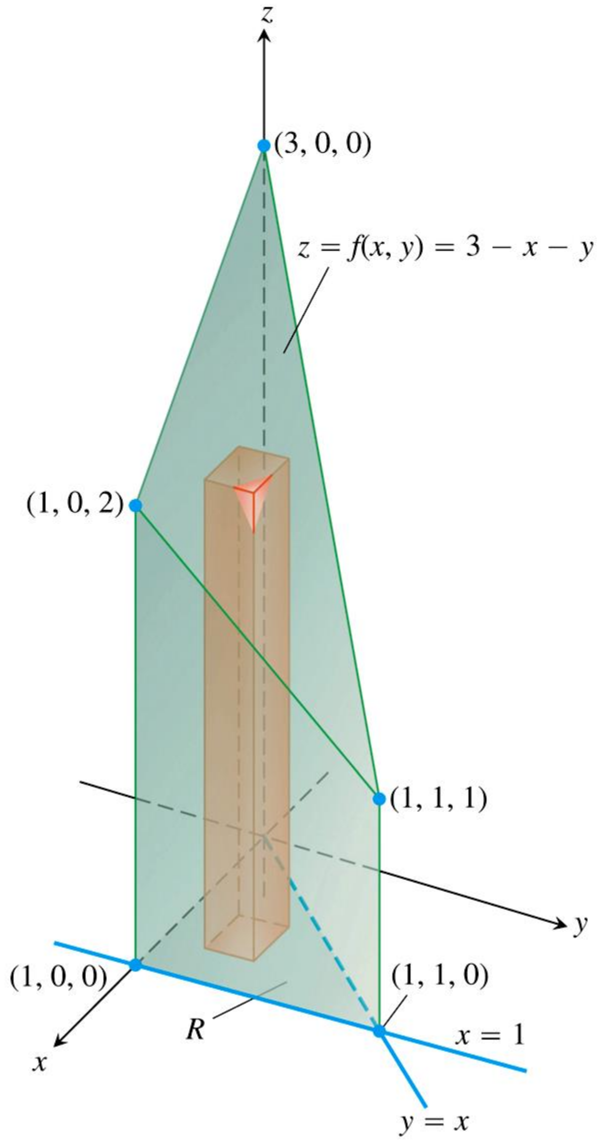
When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

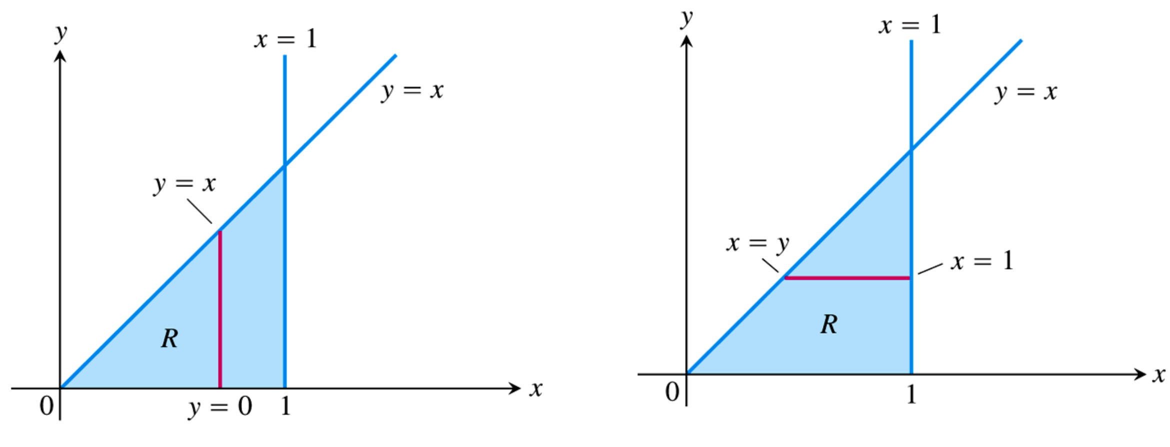
$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2} y^2 \right) dy = \left[\frac{5}{2} y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be.



(a)



(b) (c)

FIGURE 15.12 (a) Prism with a triangular base in the *xy*-plane. The volume of this prism is defined as a double integral over *R*. To evaluate it as an iterated integral, we may integrate first with respect to *y* and then with respect to *x*, or the other way around (Example 1). (b) Integration limits of

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} f(x, y)\, dy\, dx.$$

If we integrate first with respect to *y*, we integrate along a vertical line through *R* and then integrate from left to right to include all the vertical lines in *R*. (c) Integration limits of

$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} f(x, y)\, dx\, dy.$$

If we integrate first with respect to *x*, we integrate along a horizontal line through *R* and then integrate from bottom to top to include all the horizontal lines in *R*.

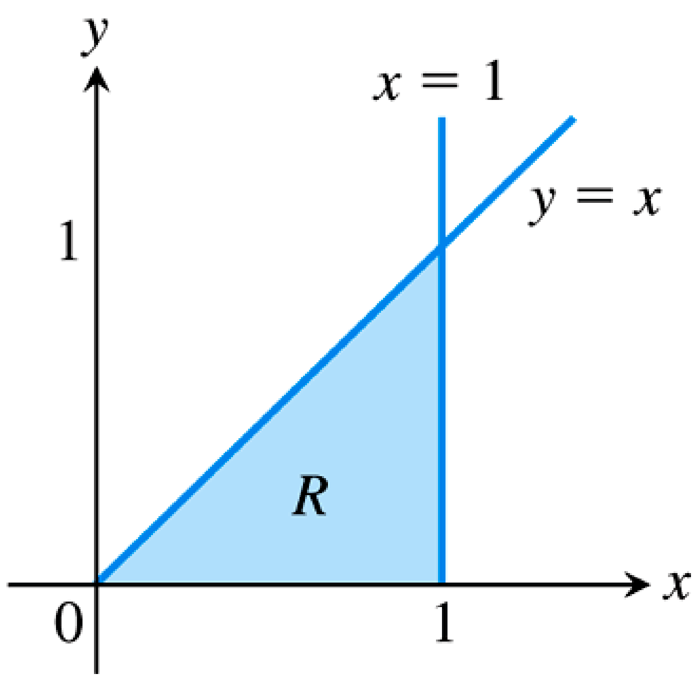


FIGURE 15.13 The region of integration in Example 2.

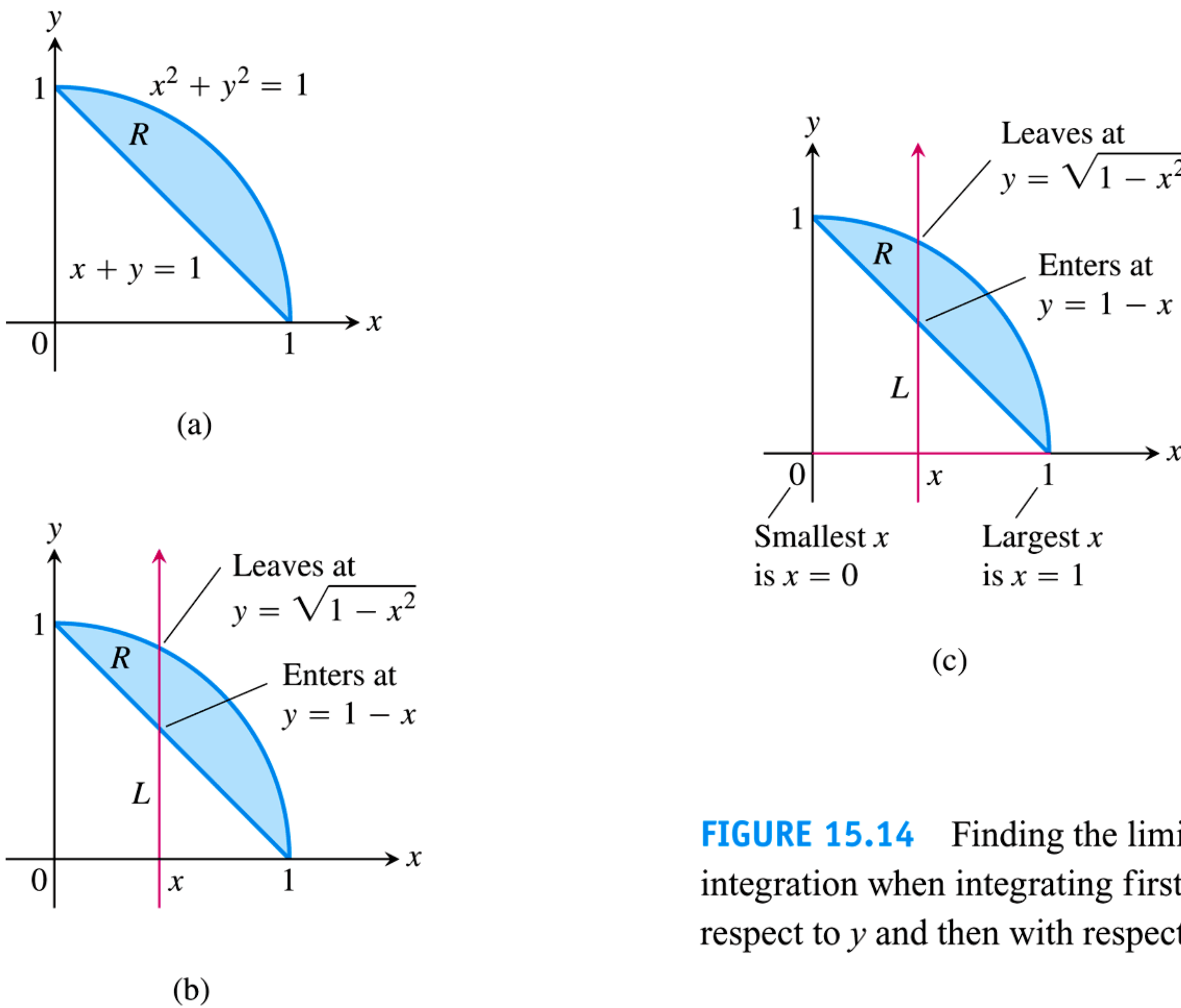


FIGURE 15.14 Finding the limits of integration when integrating first with respect to y and then with respect to x.

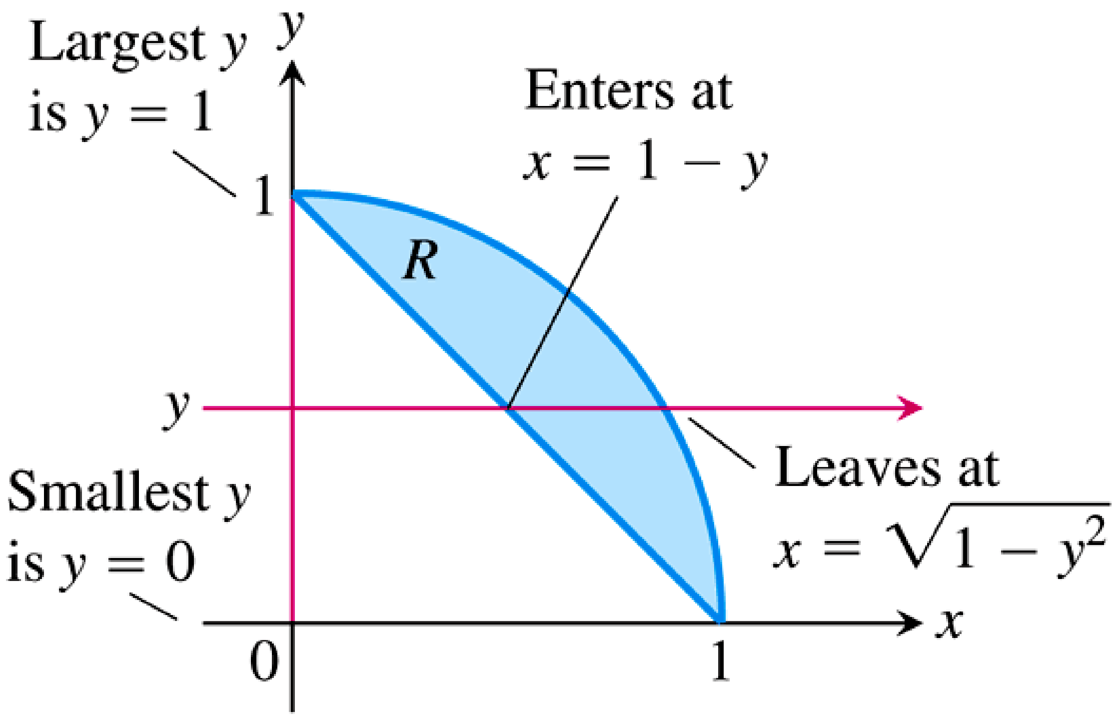


FIGURE 15.15 Finding the limits of integration when integrating first with respect to x and then with respect to y.

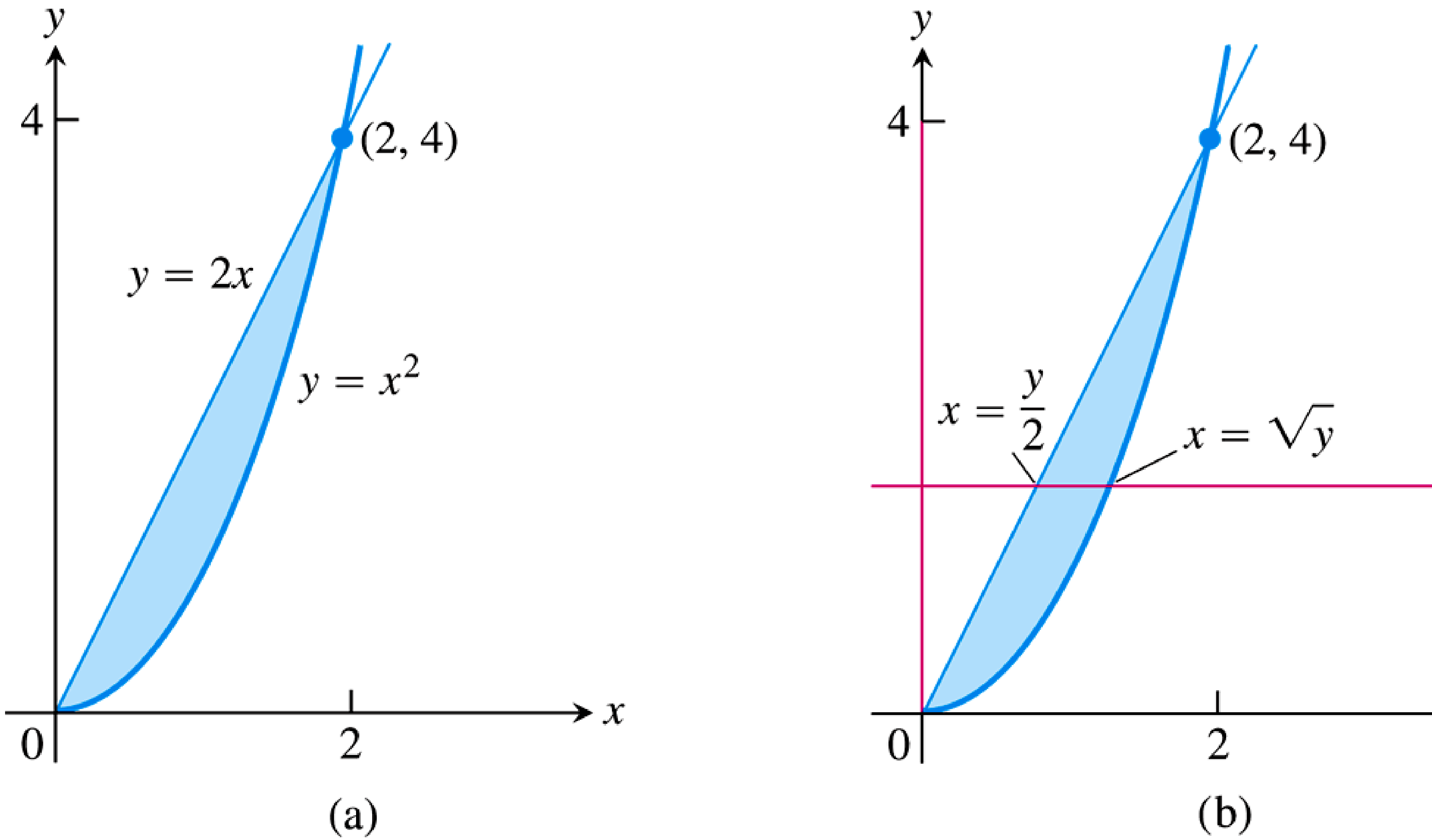


FIGURE 15.16 Region of integration for Example 3.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

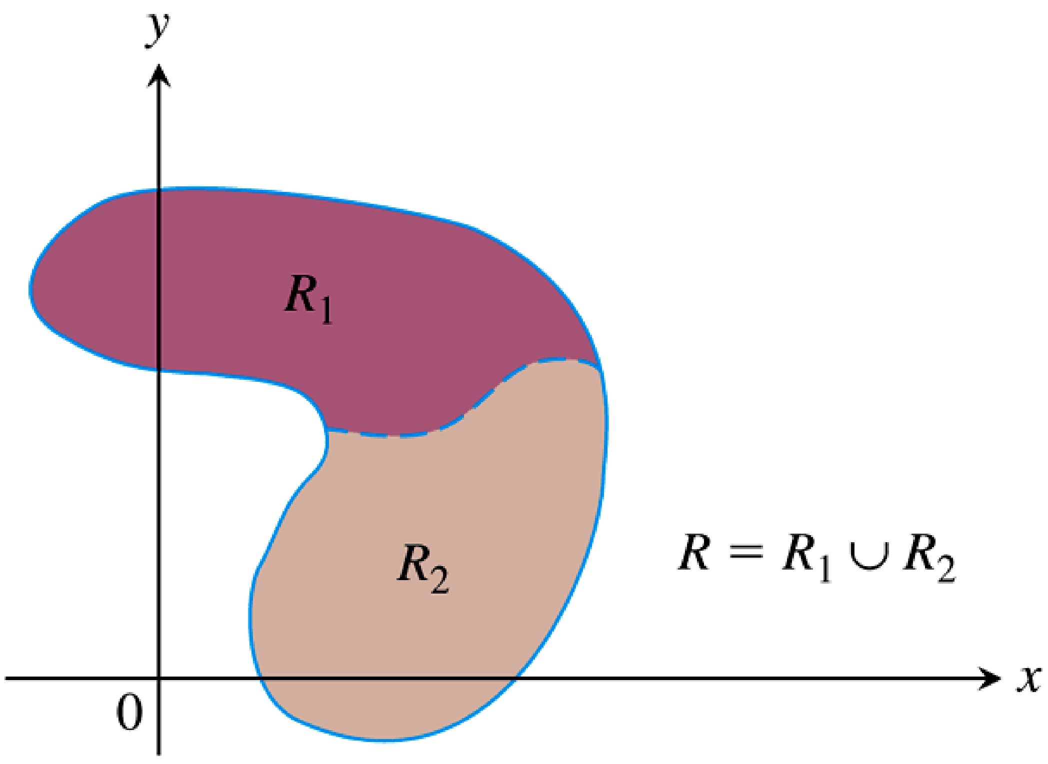
3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

if R is the union of two nonoverlapping regions R_1 and R_2



$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

FIGURE 15.17 The Additivity Property for rectangular regions holds for regions bounded by continuous curves.

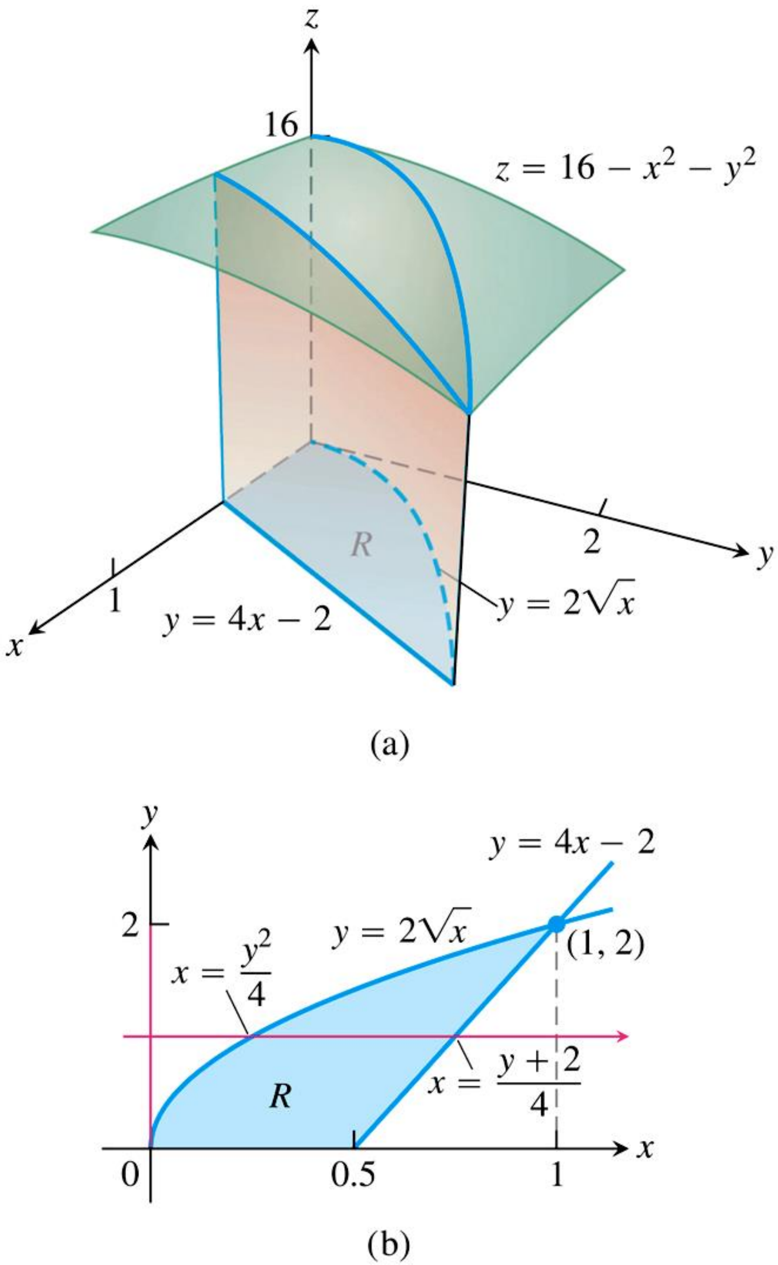


FIGURE 15.18 (a) The solid "wedgelike" region whose volume is found in Example 4. (b) The region of integration R showing the order $dx\,dy$.

15.3

Area by Double Integration

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Figure 15.19), noting where the two curves intersect at the origin and (1, 1), and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[y \Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

Notice that the single-variable integral $\int_0^1 (x - x^2)\, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.6.

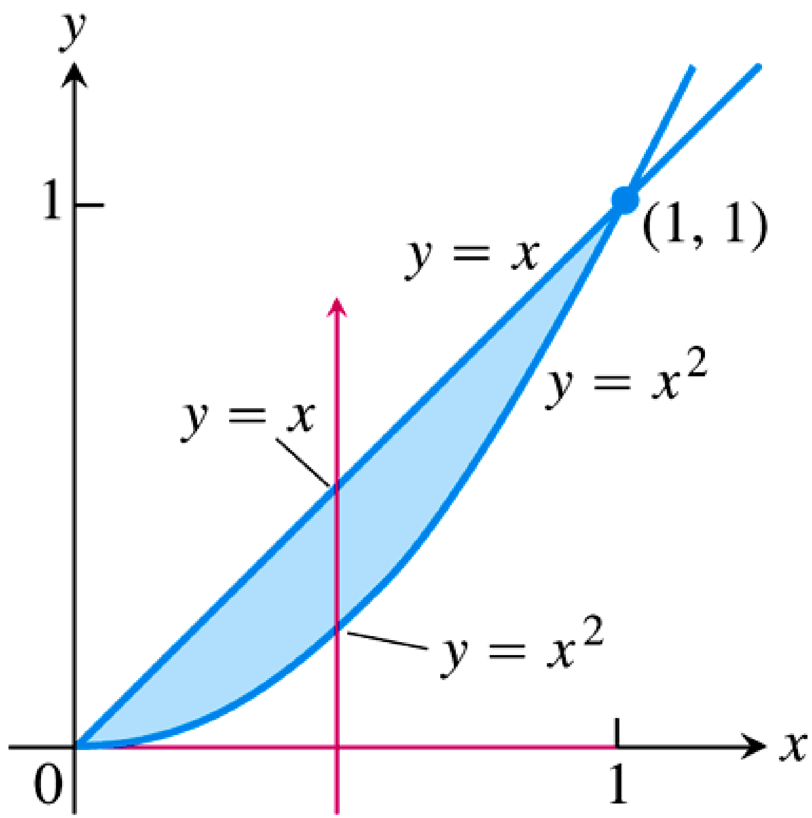
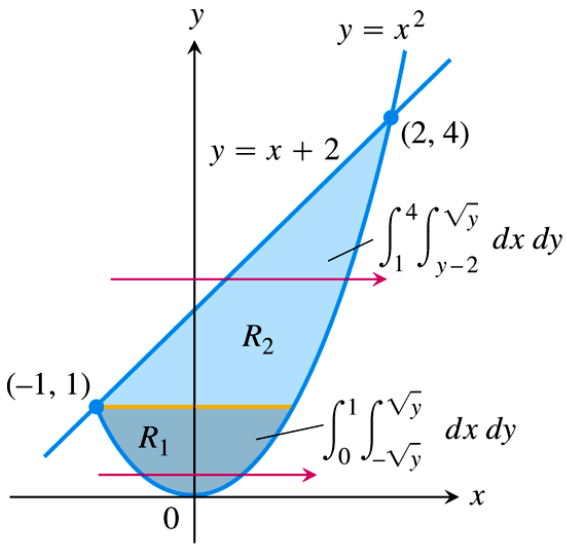
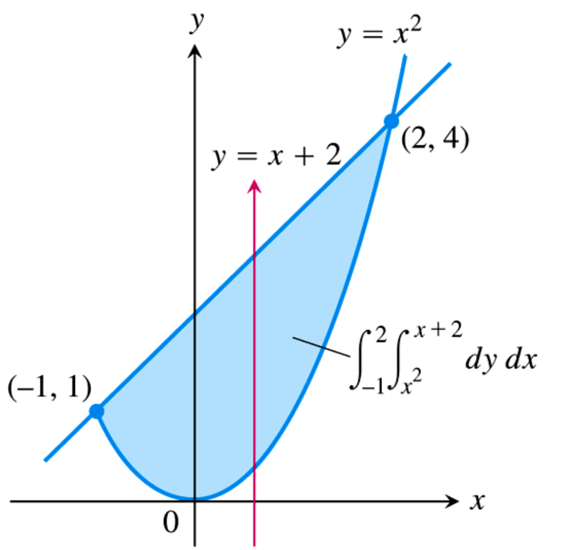


FIGURE 15.19 The region in Example 1.



(a)



(b)

FIGURE 15.20 Calculating this area takes (a) two double integrals if the first integration is with respect to x, but (b) only one if the first integration is with respect to y (Example 2).

$$\textbf{Average value} \text{ of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f \, dA. \qquad (3)$$

EXAMPLE 3 Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R: 0 \le x \le \pi,\ 0 \le y \le 1$.

Solution The value of the integral of f over R is

$$\int_0^{\pi}\int_0^1 x\cos xy\,dy\,dx = \int_0^{\pi}\Big[\sin xy\Big]_{y=0}^{y=1} dx \qquad \int x\cos xy\,dy = \sin xy + C$$

$$= \int_0^{\pi}(\sin x - 0)\,dx = -\cos x\Big]_0^{\pi} = 1 + 1 = 2.$$

The area of R is π. The average value of f over R is $2/\pi$.

15.4

Double Integrals in Polar Form

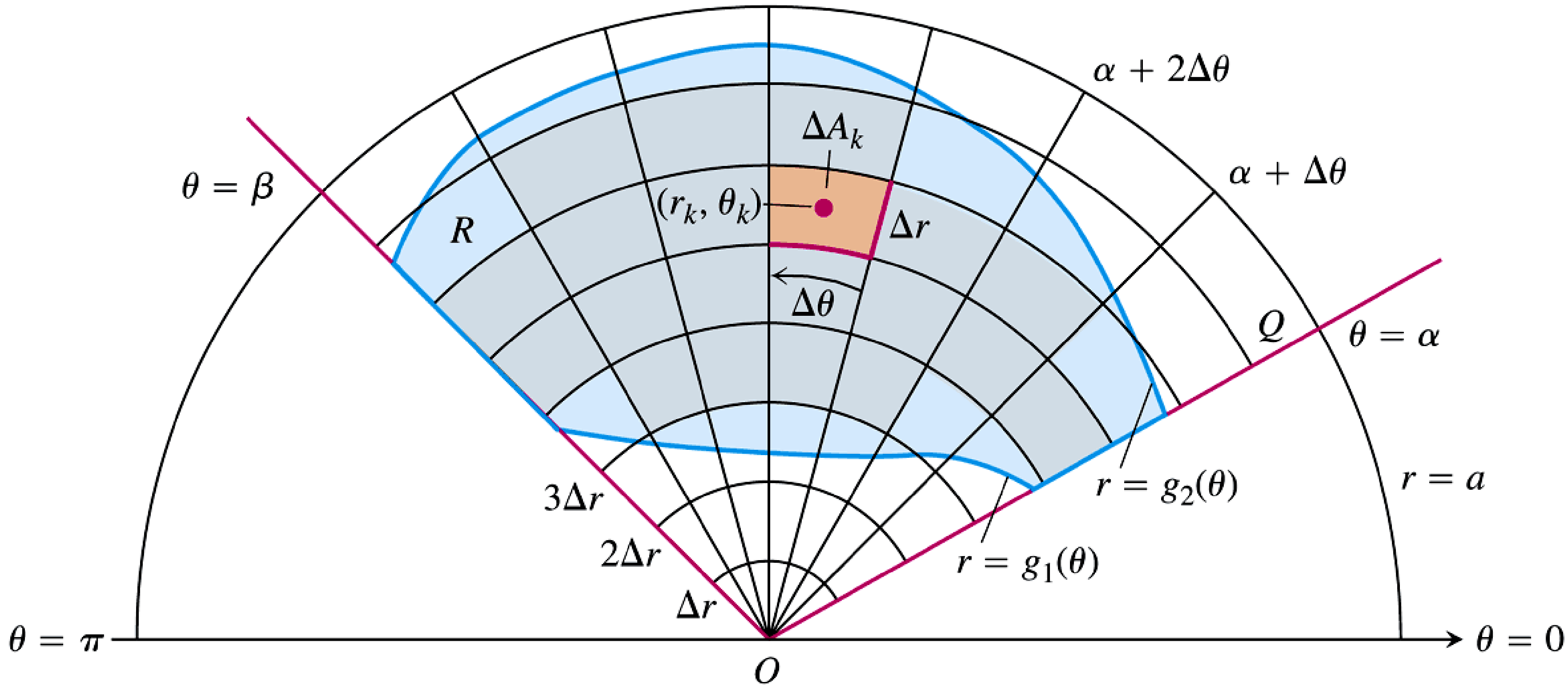


FIGURE 15.21 The region R: $g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, is contained in the fan-shaped region Q: $0 \le r \le a$, $\alpha \le \theta \le \beta$. The partition of Q by circular arcs and rays induces a partition of R.

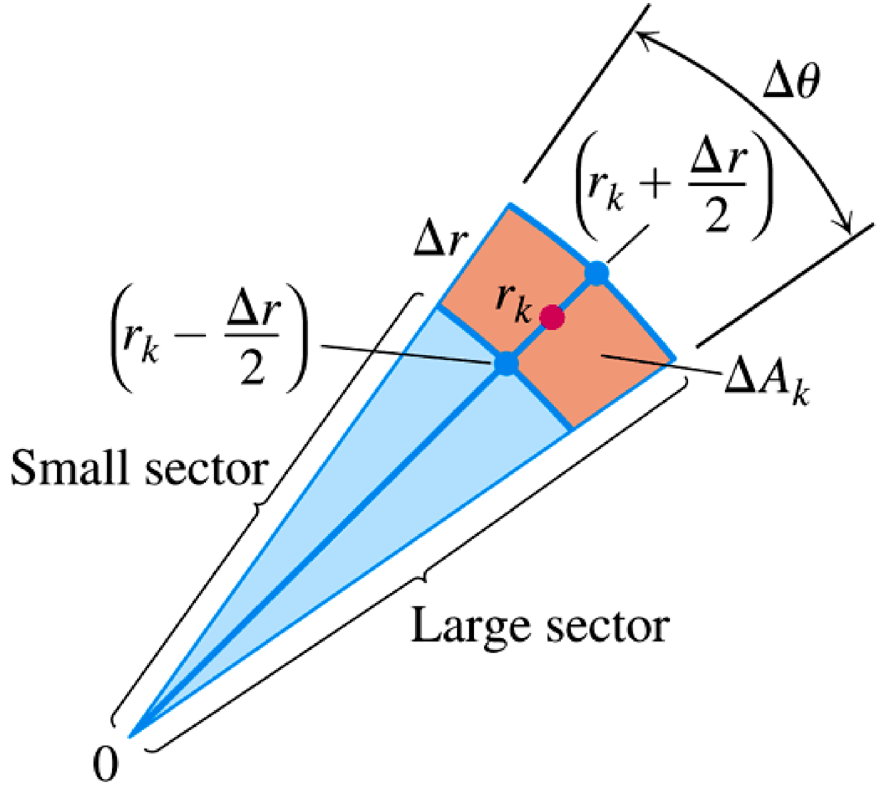


FIGURE 15.22 The observation that

$$\Delta A_k = \begin{pmatrix} \text{area of} \\ \text{large sector} \end{pmatrix} - \begin{pmatrix} \text{area of} \\ \text{small sector} \end{pmatrix}$$

leads to the formula $\Delta A_k = r_k \, \Delta r \, \Delta \theta$.

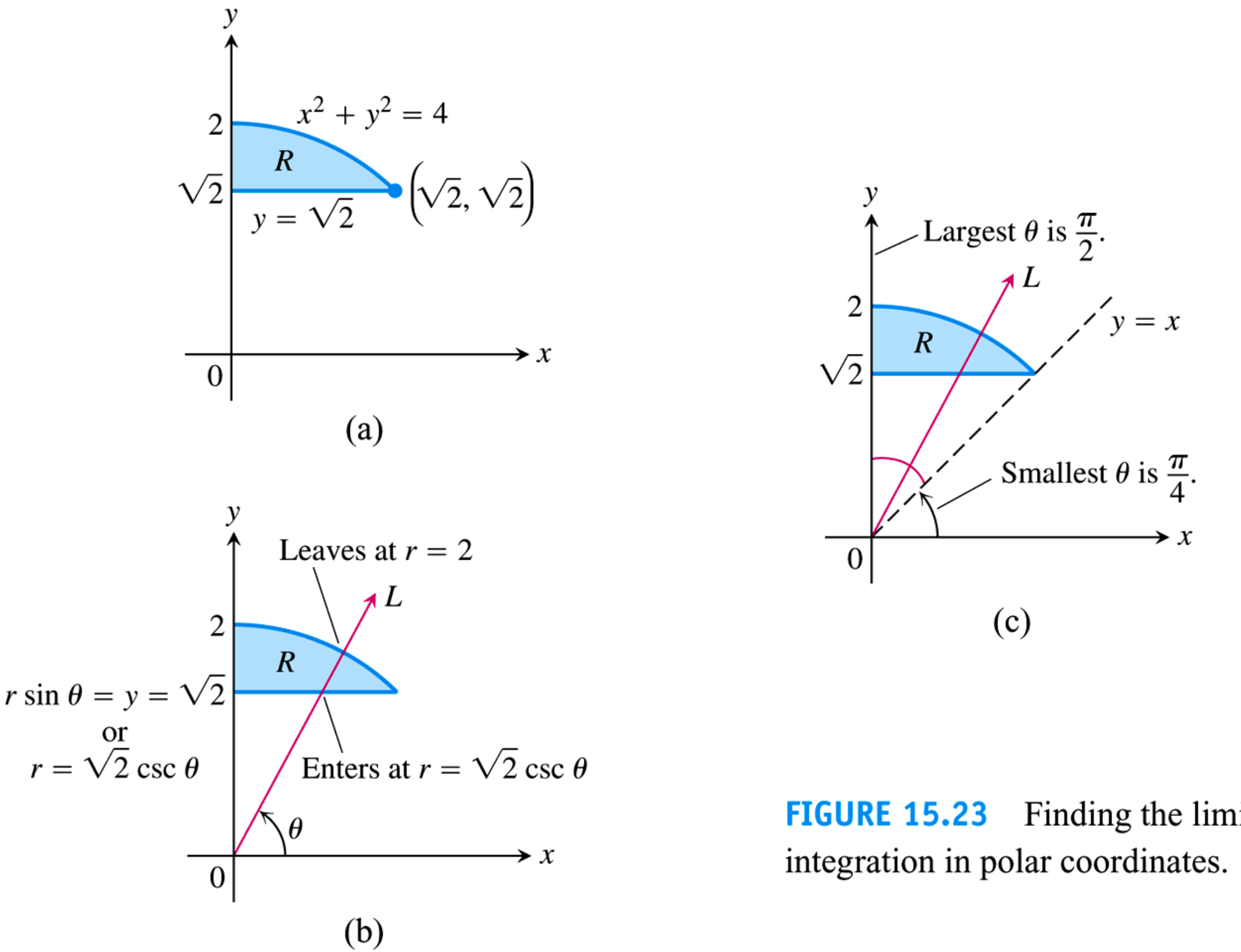


FIGURE 15.23 Finding the limits of integration in polar coordinates.

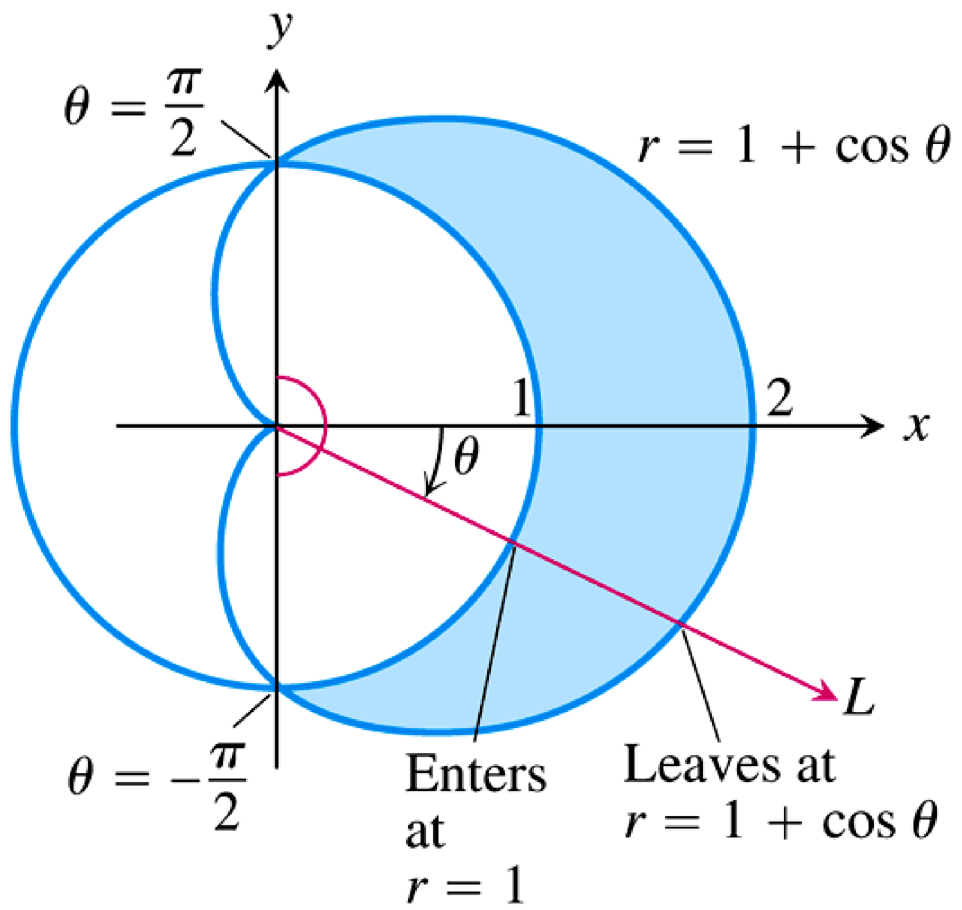


FIGURE 15.24 Finding the limits of integration in polar coordinates for the region in Example 1.

EXAMPLE 1 Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves (Figure 15.24).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.
3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta.$$

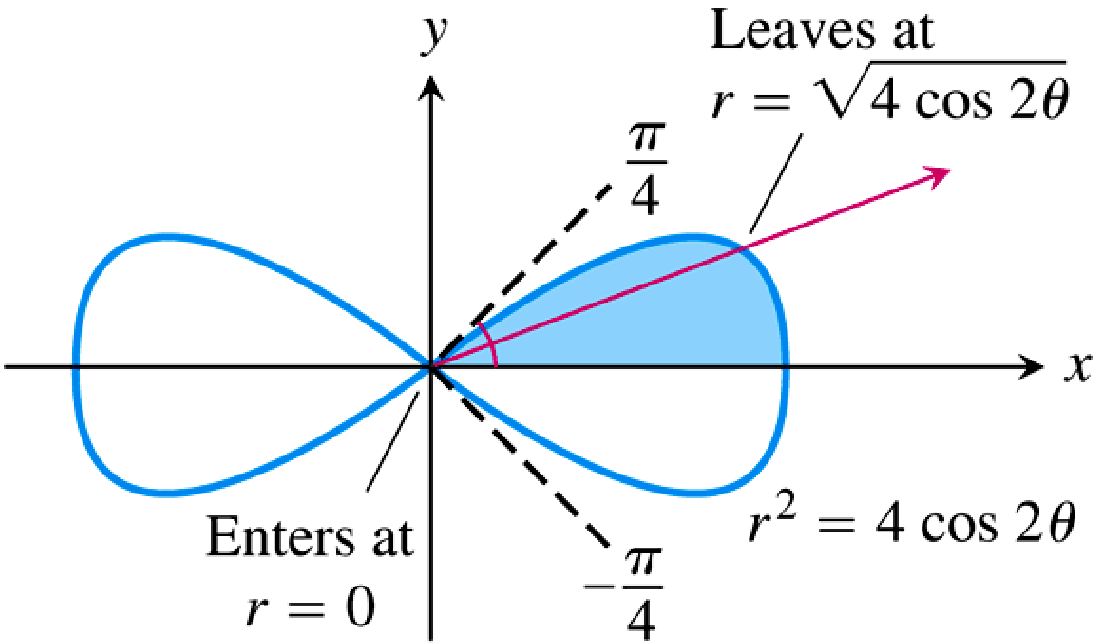


FIGURE 15.25 To integrate over the shaded region, we run r from 0 to $\sqrt{4 \cos 2\theta}$ and θ from 0 to $\pi/4$ (Example 2).

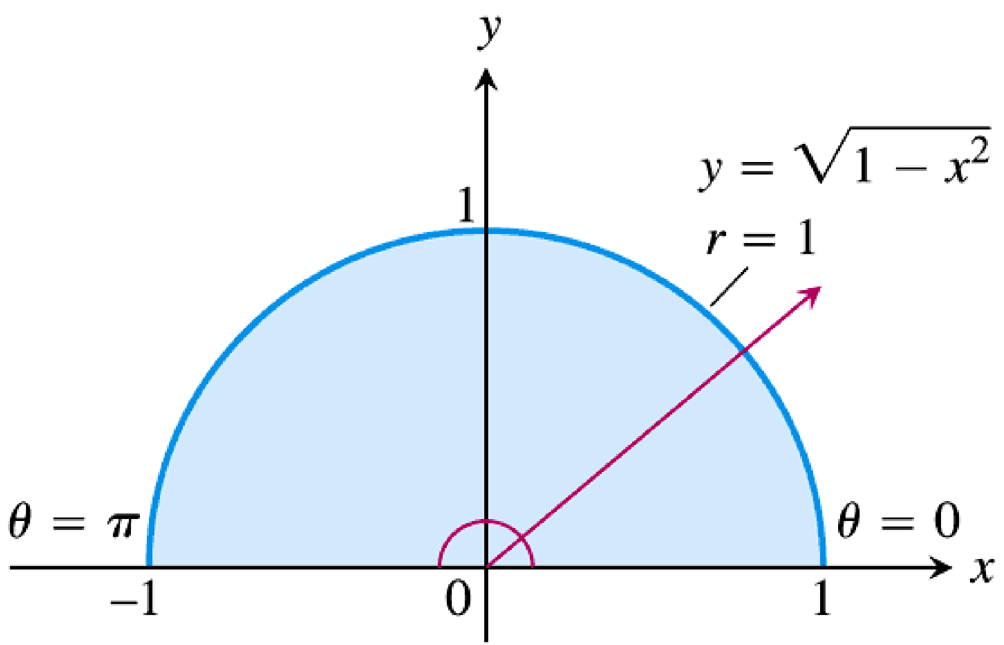


FIGURE 15.26 The semicircular region in Example 4 is the region

$$0 \leq r \leq 1, \qquad 0 \leq \theta \leq \pi.$$

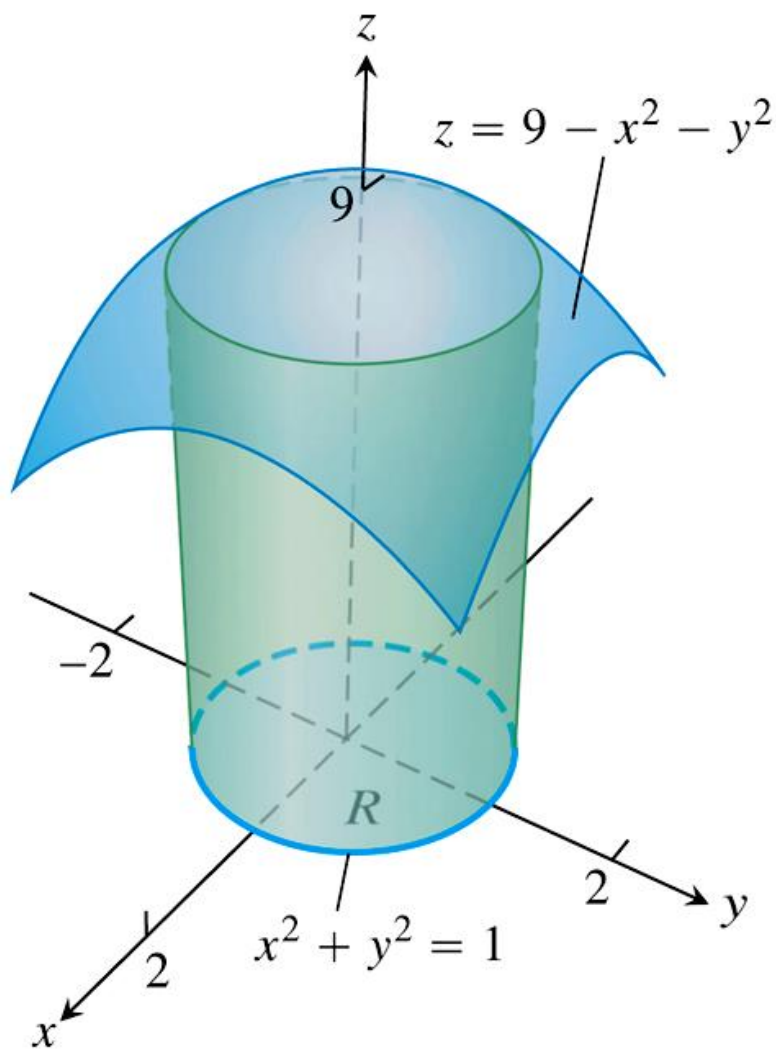


FIGURE 15.27 The solid region in Example 5.

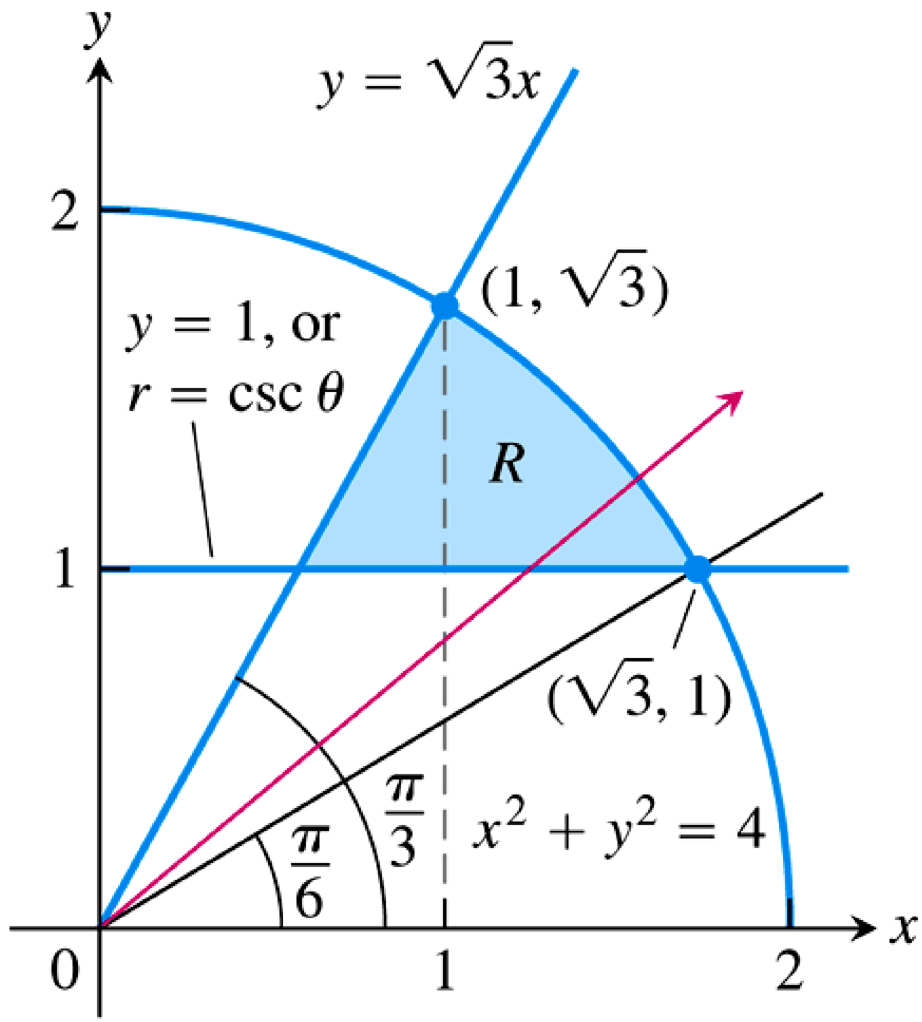


FIGURE 15.28 The region R in Example 6.

15.5

Triple Integrals in Rectangular Coordinates

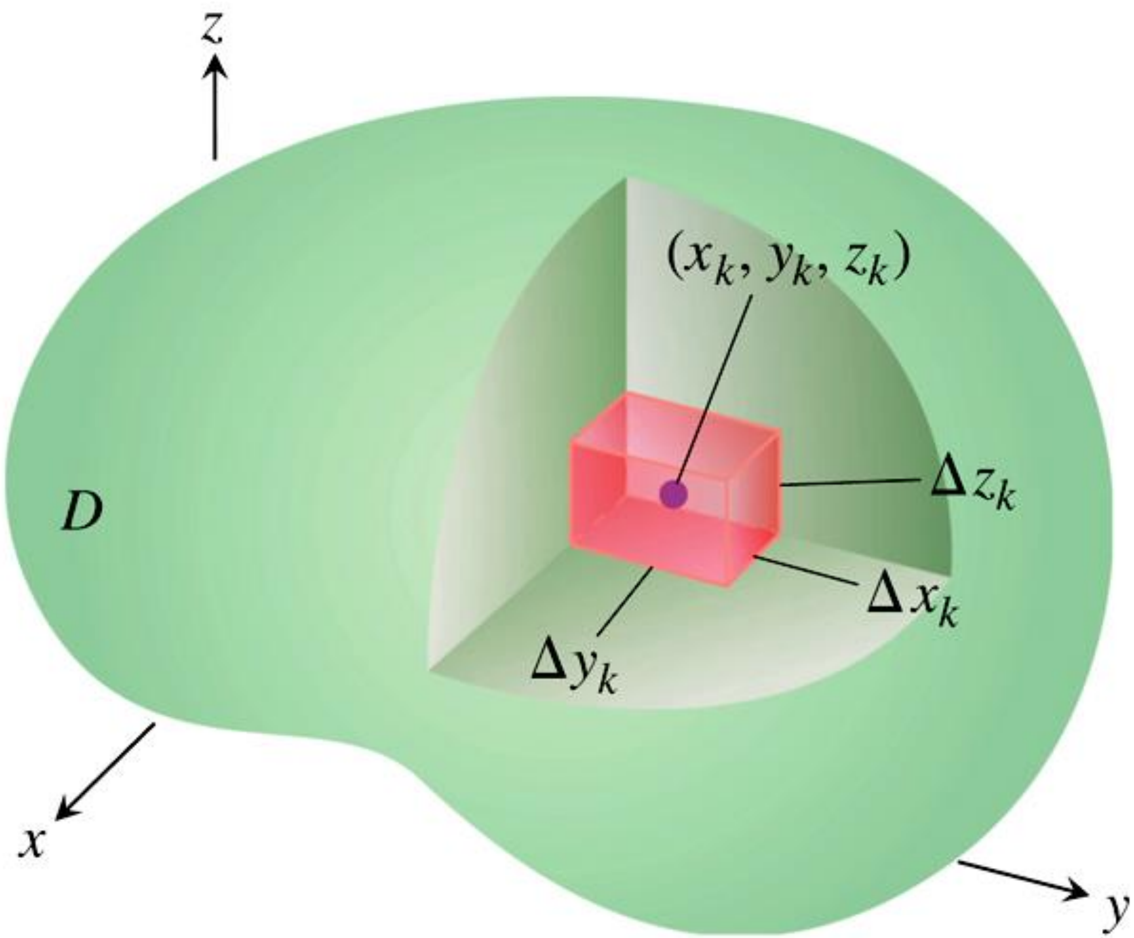


FIGURE 15.29 Partitioning a solid with rectangular cells of volume ΔV_k.

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

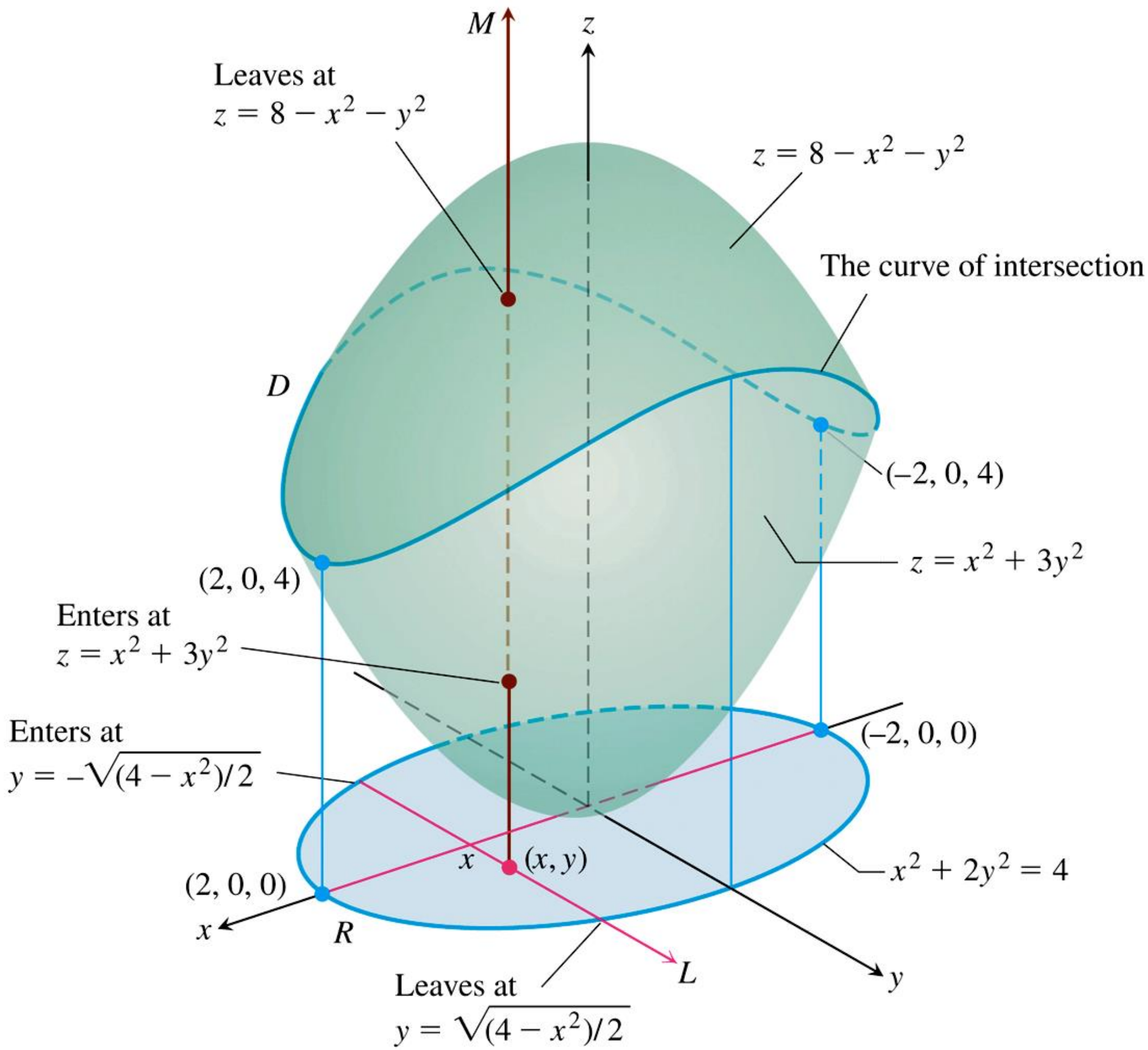


FIGURE 15.30 The volume of the region enclosed by two paraboloids, calculated in Example 1.

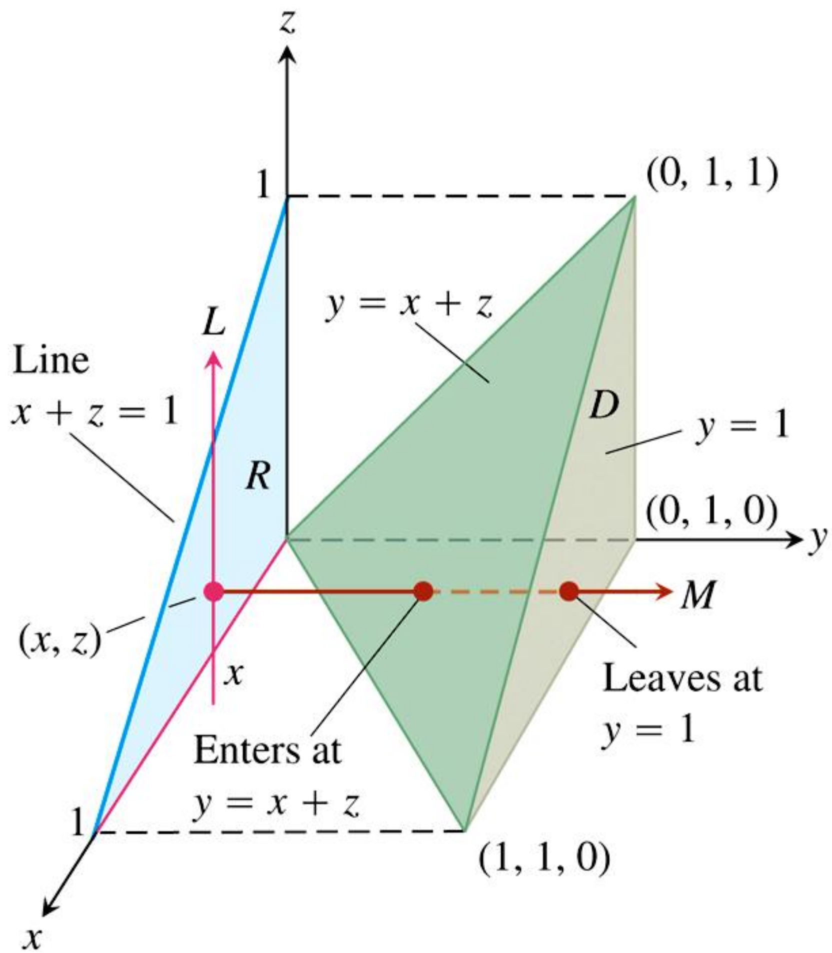


FIGURE 15.31 Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron D (Examples 2 and 3).

EXAMPLE 3 Integrate $F(x, y, z) = 1$ over the tetrahedron D in Example 2 in the order $dz\,dy\,dx$, and then integrate in the order $dy\,dz\,dx$.

Solution First we find the z-limits of integration. A line M parallel to the z-axis through a typical point (x, y) in the xy-plane "shadow" enters the tetrahedron at $z = 0$ and exits through the upper plane where $z = y - x$ (Figure 15.32).

Next we find the y-limits of integration. On the xy-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line L through (x, y) parallel to the y-axis enters the shadow in the xy-plane at $y = x$ and exits at $y = 1$ (Figure 15.32).

Finally we find the x-limits of integration. As the line L parallel to the y-axis in the previous step sweeps out the shadow, the value of x varies from $x = 0$ to $x = 1$ at the point $(1, 1, 0)$ (see Figure 15.32). The integral is

$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\,dz\,dy\,dx.$$

For example, if $F(x, y, z) = 1$, we would find the volume of the tetrahedron to be

$$\begin{aligned} V &= \int_0^1 \int_x^1 \int_0^{y-x} dz\,dy\,dx \\ &= \int_0^1 \int_x^1 (y - x)\,dy\,dx \\ &= \int_0^1 \left[\frac{1}{2}y^2 - xy\right]_{y=x}^{y=1} dx \\ &= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3\right]_0^1 \\ &= \frac{1}{6}. \end{aligned}$$

Continued on next page

We get the same result by integrating with the order *dy dz dx*. From Example 2,

$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-x} \int_{x+z}^{1} dy\, dz\, dx \\
&= \int_0^1 \int_0^{1-x} (1 - x - z)\, dz\, dx \\
&= \int_0^1 \left[(1 - x)z - \frac{1}{2} z^2 \right]_{z=0}^{z=1-x} dx \\
&= \int_0^1 \left[(1 - x)^2 - \frac{1}{2} (1 - x)^2 \right] dx \\
&= \frac{1}{2} \int_0^1 (1 - x)^2\, dx \\
&= -\frac{1}{6} (1 - x)^3 \Big]_0^1 = \frac{1}{6}.
\end{aligned}
$$

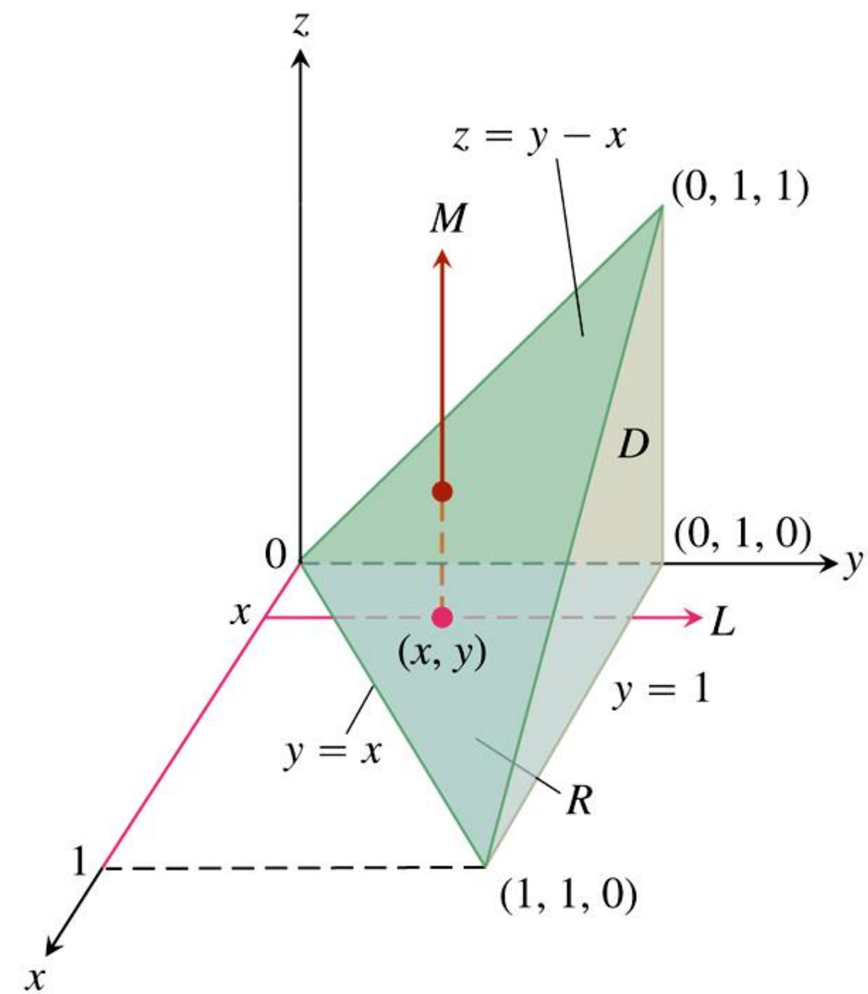


FIGURE 15.32 The tetrahedron in Example 3 showing how the limits of integration are found for the order $dz\,dy\,dx$.

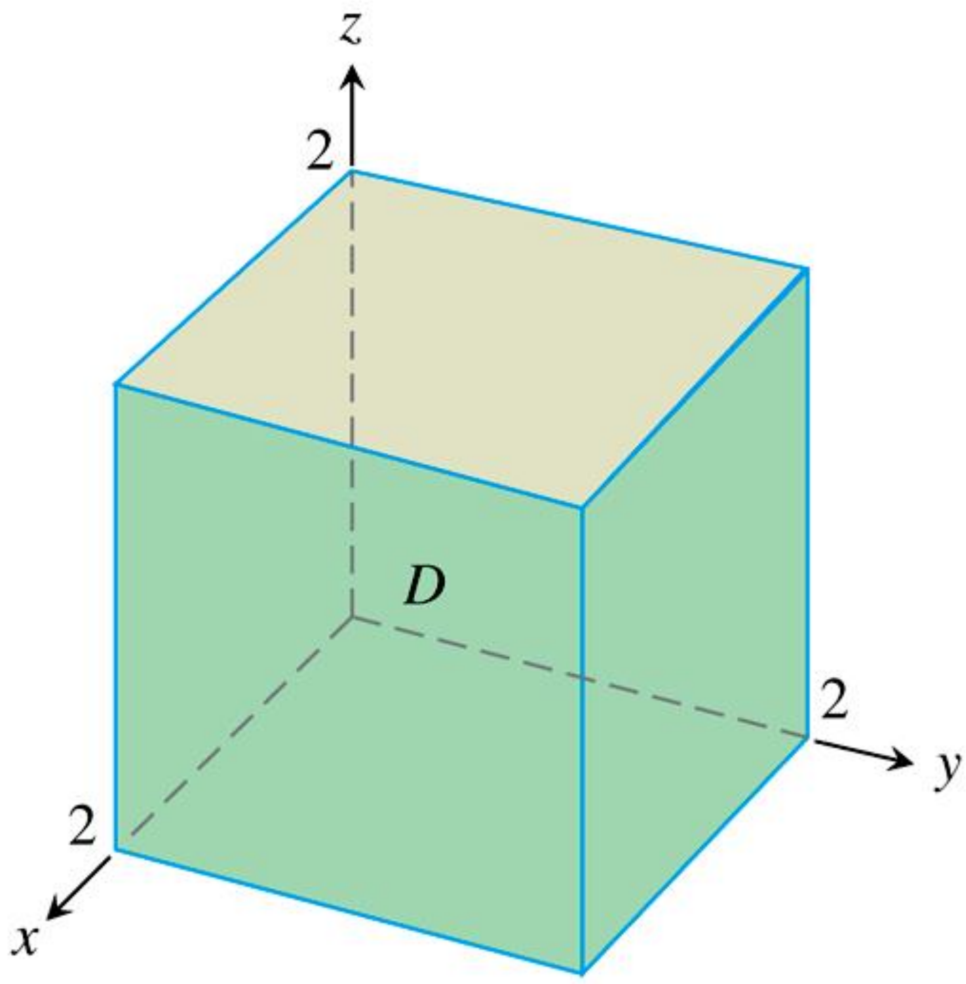


FIGURE 15.33 The region of integration in Example 4.

15.6

Moments and Centers of Mass

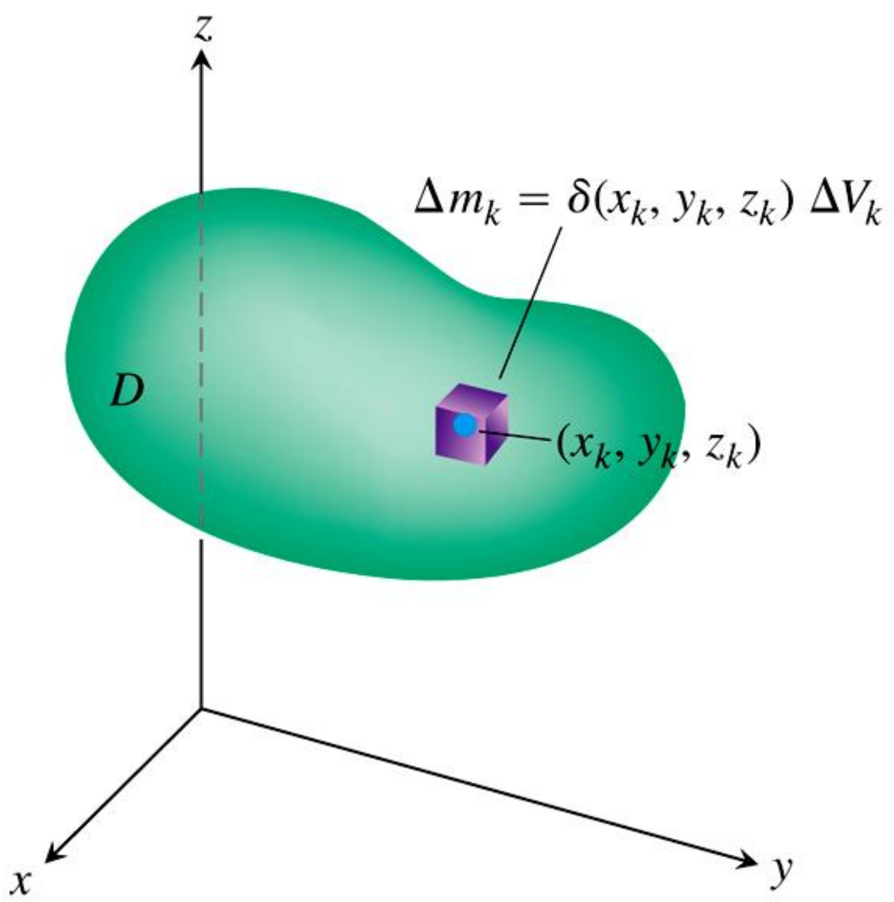


FIGURE 15.34 To define an object's mass, we first imagine it to be partitioned into a finite number of mass elements Δm_k.

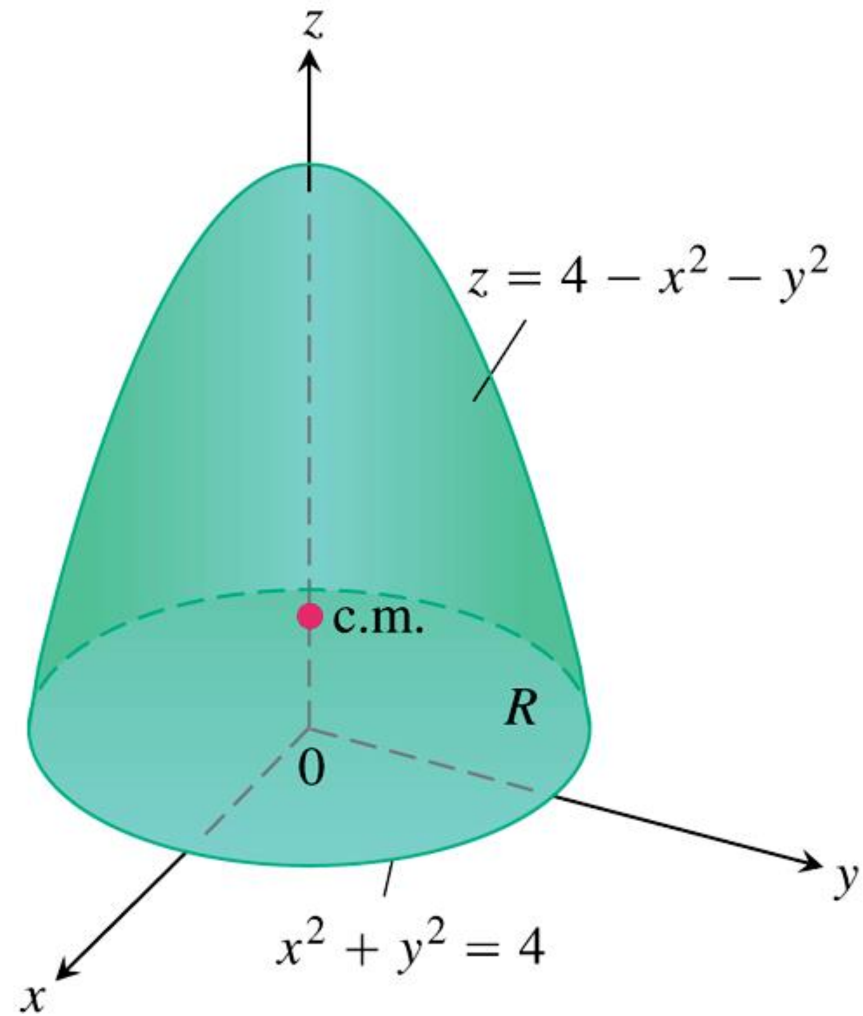


FIGURE 15.35 Finding the center of mass of a solid (Example 1).

EXAMPLE 1 Find the center of mass of a solid of constant density δ bounded below by the disk R: $x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Figure 15.35).

Solution By symmetry $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we first calculate

$$M_{xy} = \iint_R \int_{z=0}^{z=4-x^2-y^2} z\,\delta\,dz\,dy\,dx = \iint_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\,dy\,dx$$

$$= \frac{\delta}{2}\iint_R (4 - x^2 - y^2)^2\,dy\,dx$$

$$= \frac{\delta}{2}\int_0^{2\pi}\int_0^2 (4 - r^2)^2\,r\,dr\,d\theta \qquad \text{Polar coordinates simplify the integration.}$$

$$= \frac{\delta}{2}\int_0^{2\pi}\left[-\frac{1}{6}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3}\int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.$$

A similar calculation gives the mass

$$M = \iint_R \int_0^{4-x^2-y^2} \delta\,dz\,dy\,dx = 8\pi\delta.$$

Therefore $\bar{z} = (M_{xy}/M) = 4/3$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$.

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint\limits_D \delta \, dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_D x \, \delta \, dV, \qquad M_{xz} = \iiint\limits_D y \, \delta \, dV, \qquad M_{xy} = \iiint\limits_D z \, \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint\limits_R \delta \, dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint\limits_R x \, \delta \, dA, \qquad M_x = \iint\limits_R y \, \delta \, dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$

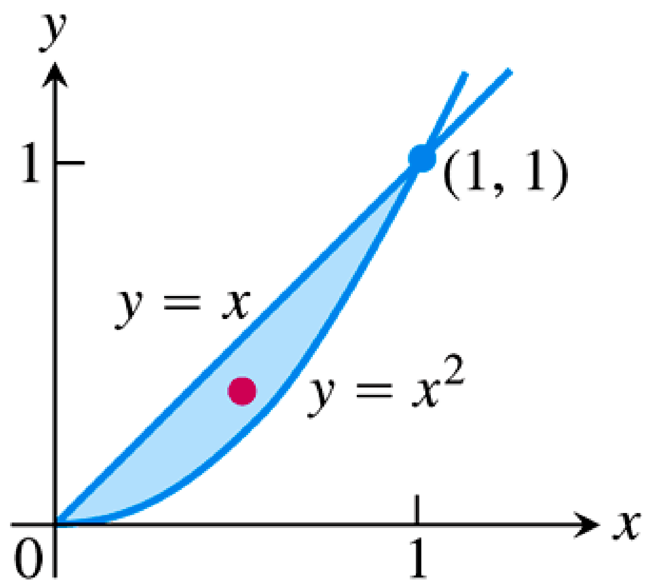


FIGURE 15.36 The centroid of this region is found in Example 2.

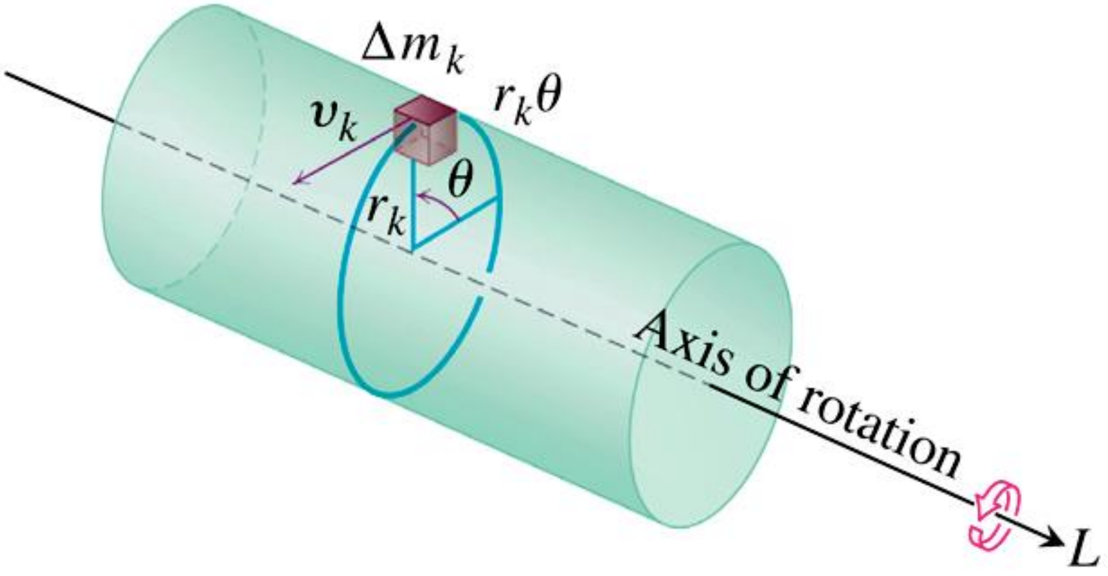


FIGURE 15.37 To find an integral for the amount of energy stored in a rotating shaft, we first imagine the shaft to be partitioned into small blocks. Each block has its own kinetic energy. We add the contributions of the individual blocks to find the kinetic energy of the shaft.

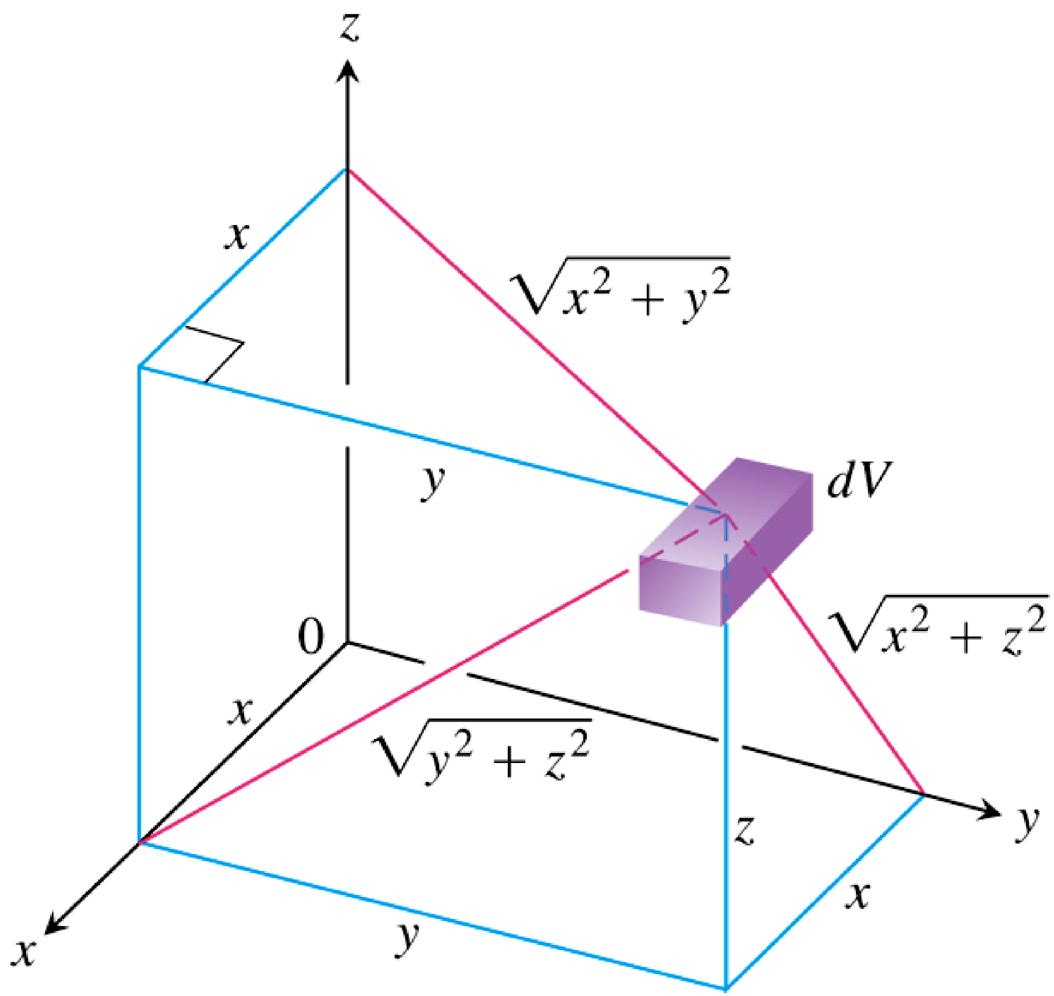


FIGURE 15.38 Distances from dV to the coordinate planes and axes.

EXAMPLE 3 Find I_x, I_y, I_z for the rectangular solid of constant density δ shown in Figure 15.39.

Solution The formula for I_x gives

$$I_x = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2)\, \delta\, dx\, dy\, dz.$$

We can avoid some of the work of integration by observing that $(y^2 + z^2)\delta$ is an even function of x, y, and z since δ is constant. The rectangular solid consists of eight symmetric pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value.

$$\begin{aligned}
I_x &= 8\int_0^{c/2} \int_0^{b/2} \int_0^{a/2} (y^2 + z^2)\, \delta\, dx\, dy\, dz = 4a\delta \int_0^{c/2} \int_0^{b/2} (y^2 + z^2)\, dy\, dz \\
&= 4a\delta \int_0^{c/2} \left[\frac{y^3}{3} + z^2 y\right]_{y=0}^{y=b/2} dz \\
&= 4a\delta \int_0^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz \\
&= 4a\delta\left(\frac{b^3 c}{48} + \frac{c^3 b}{48}\right) = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2). \qquad M = abc\delta
\end{aligned}$$

Similarly,

$$I_y = \frac{M}{12}(a^2 + c^2) \qquad \text{and} \qquad I_z = \frac{M}{12}(a^2 + b^2).$$

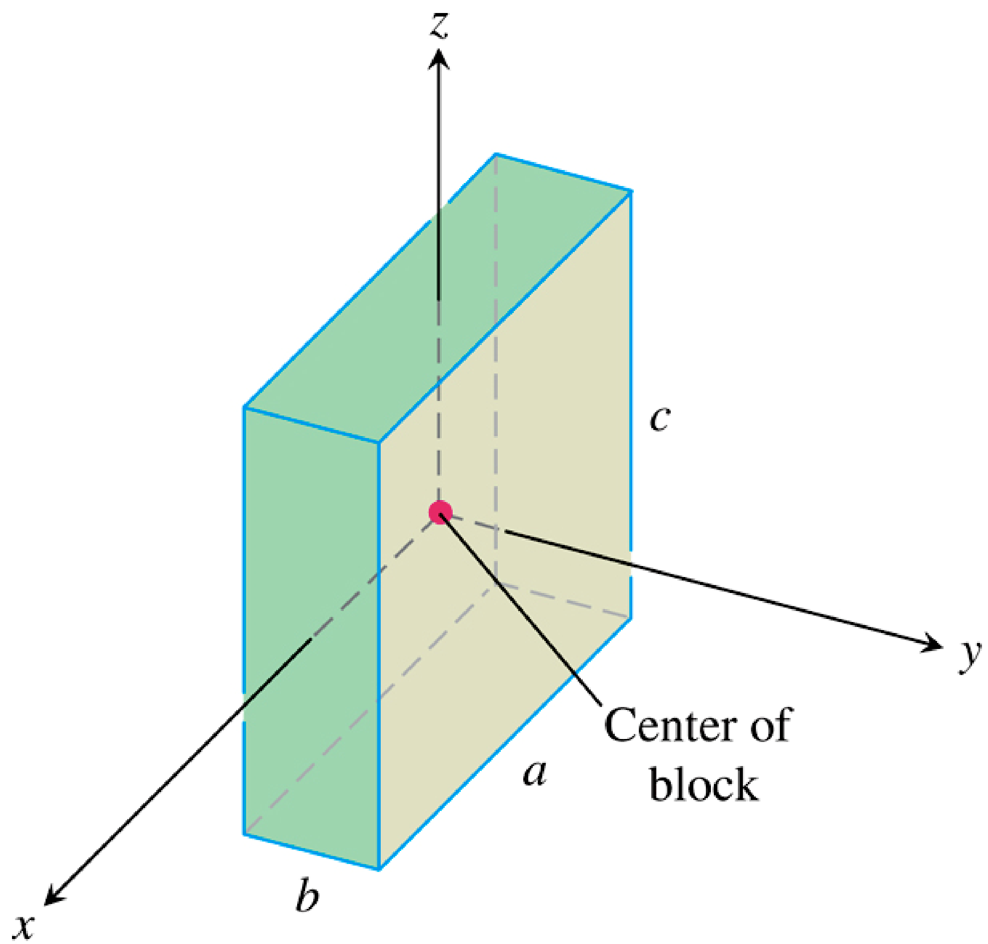


FIGURE 15.39 Finding I_x, I_y, and I_z for the block shown here. The origin lies at the center of the block (Example 3).

TABLE 15.2 Moments of inertia (second moments) formulas

THREE-DIMENSIONAL SOLID		
About the x-axis:	$I_x = \iiint (y^2 + z^2)\,\delta\, dV$	$\delta = \delta(x, y, z)$
About the y-axis:	$I_y = \iiint (x^2 + z^2)\,\delta\, dV$	
About the z-axis:	$I_z = \iiint (x^2 + y^2)\,\delta\, dV$	
About a line L:	$I_L = \iiint r^2\,\delta\, dV$	$r(x, y, z) =$ distance from the point (x, y, z) to line L
TWO-DIMENSIONAL PLATE		
About the x-axis:	$I_x = \iint y^2\,\delta\, dA$	$\delta = \delta(x, y)$
About the y-axis:	$I_y = \iint x^2\,\delta\, dA$	
About a line L:	$I_L = \iint r^2(x, y)\,\delta\, dA$	$r(x, y) =$ distance from (x, y) to L
About the origin (polar moment):	$I_0 = \iint (x^2 + y^2)\,\delta\, dA = I_x + I_y$	

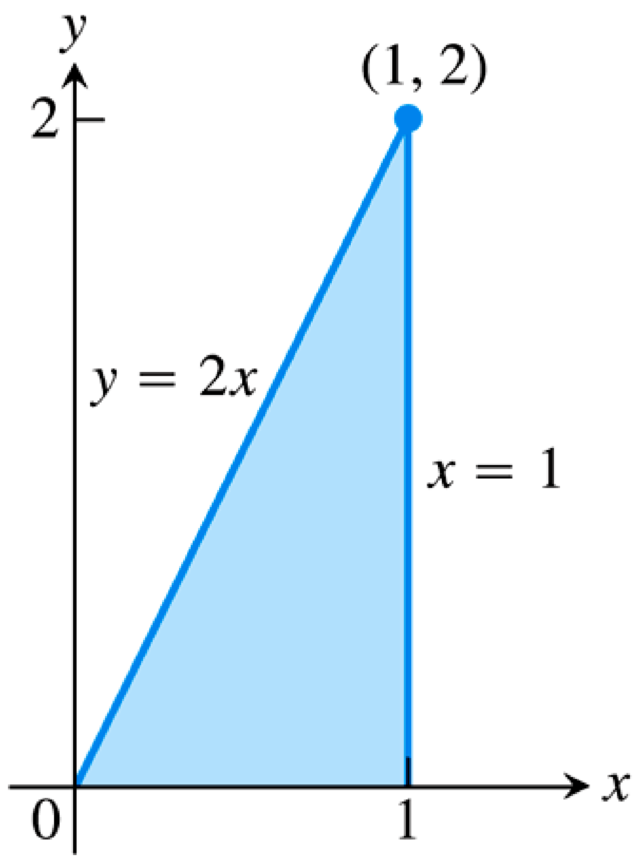


FIGURE 15.40 The triangular region covered by the plate in Example 4.

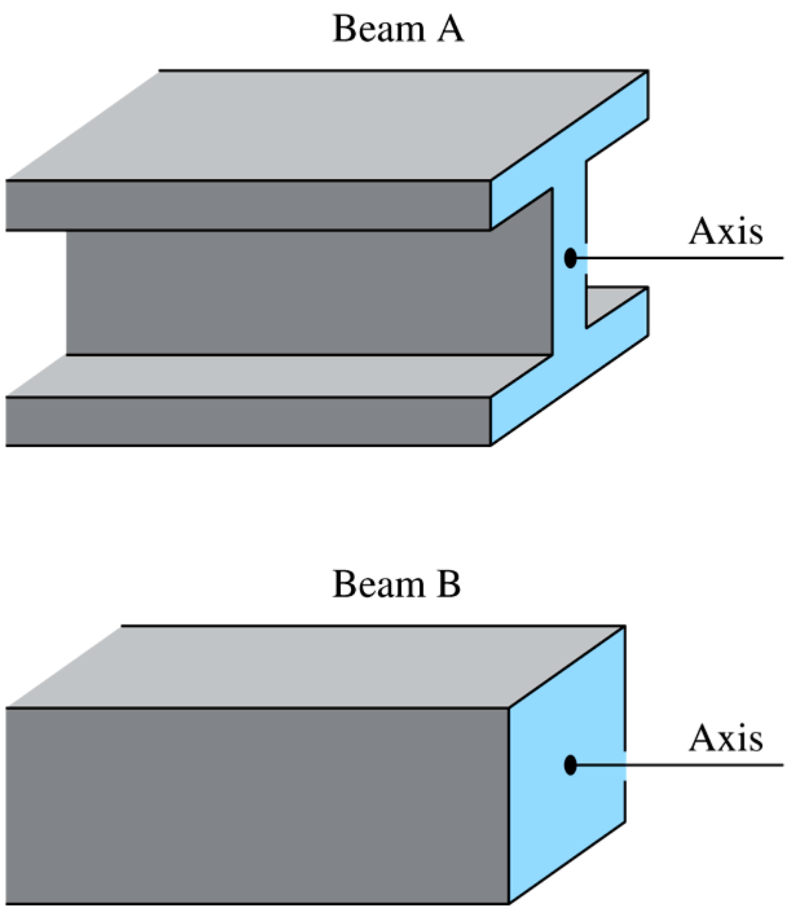


FIGURE 15.41 The greater the polar moment of inertia of the cross-section of a beam about the beam's longitudinal axis, the stiffer the beam. Beams A and B have the same cross-sectional area, but A is stiffer.

15.7

Triple Integrals in Cylindrical and Spherical Coordinates

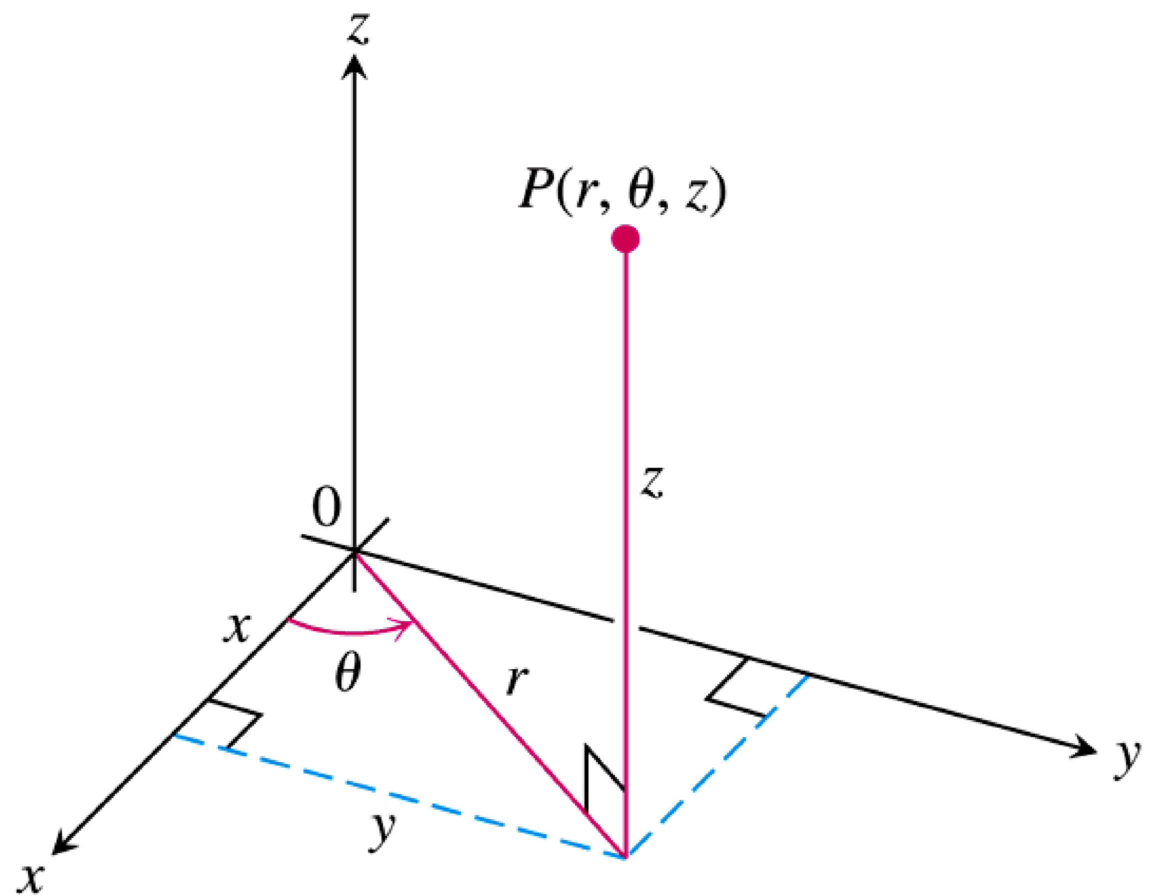


FIGURE 15.42 The cylindrical coordinates of a point in space are r, θ, and z.

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

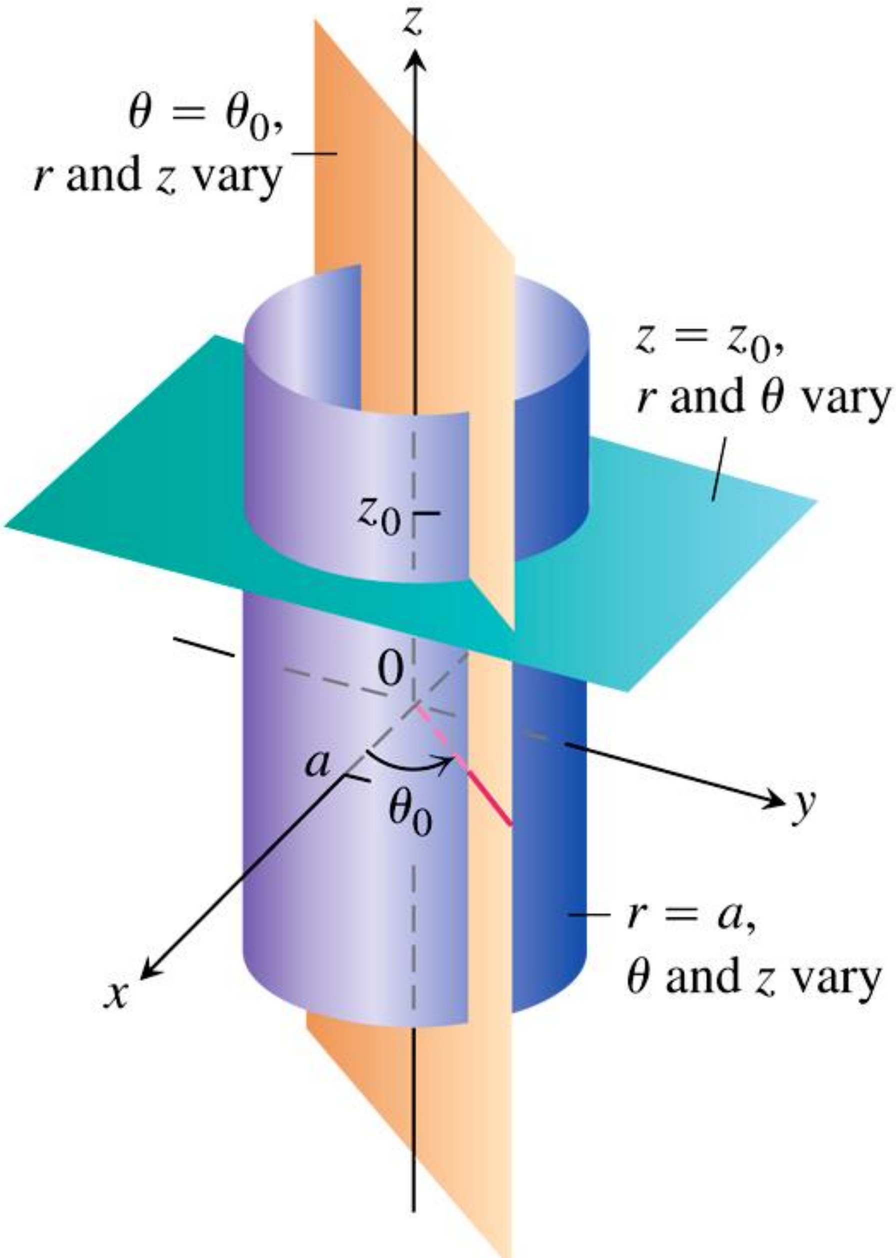


FIGURE 15.43 Constant-coordinate equations in cylindrical coordinates yield cylinders and planes.

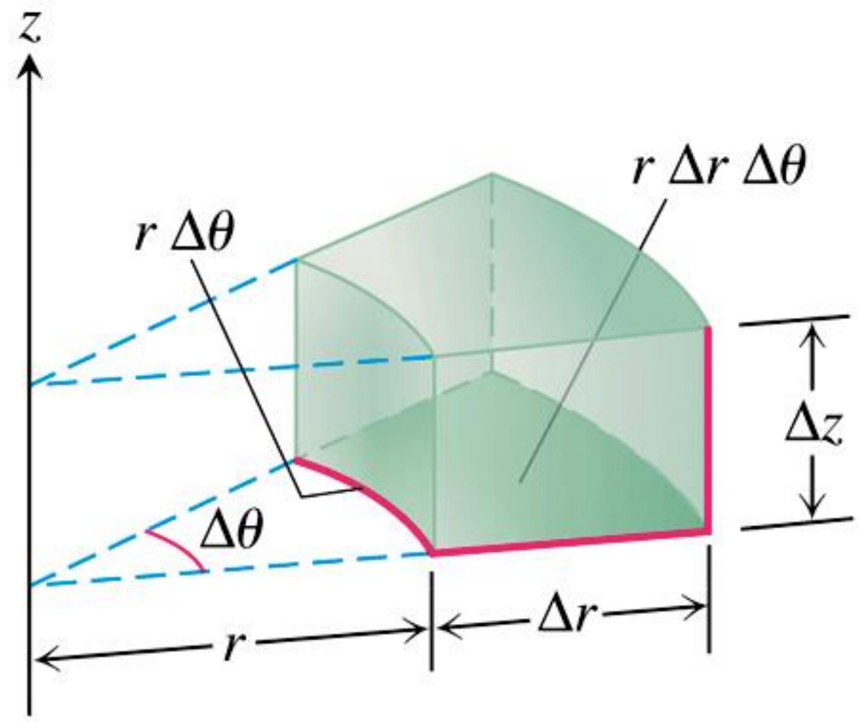


FIGURE 15.44 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$.

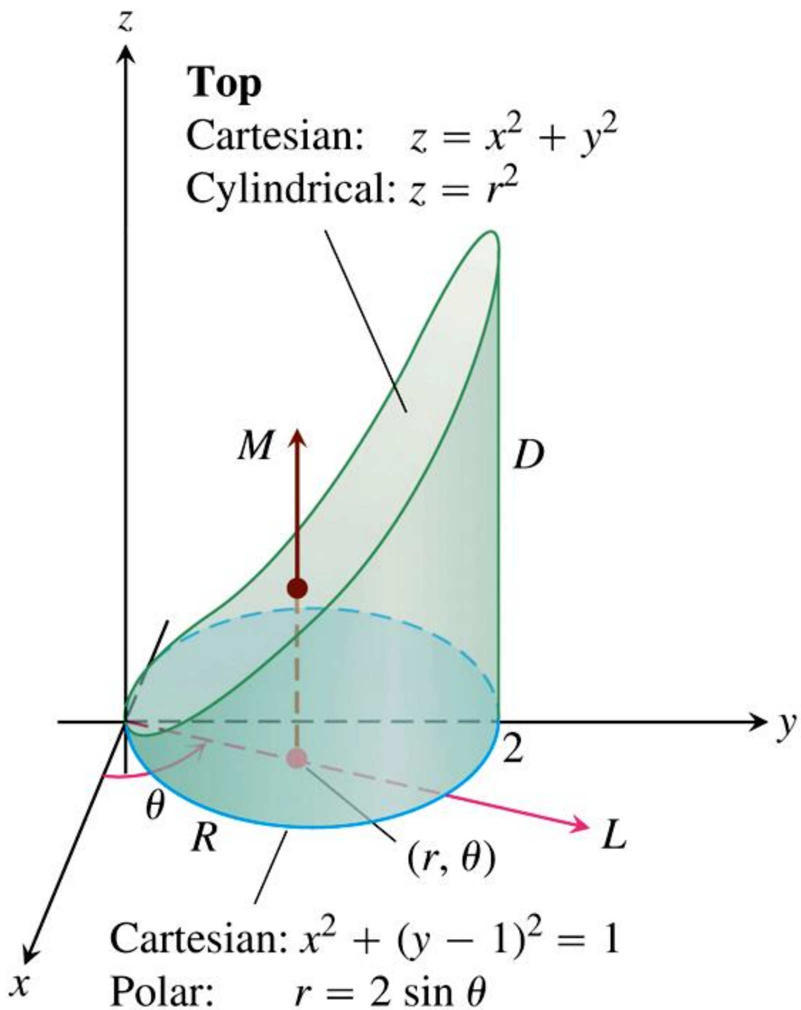


FIGURE 15.45 Finding the limits of integration for evaluating an integral in cylindrical coordinates (Example 1).

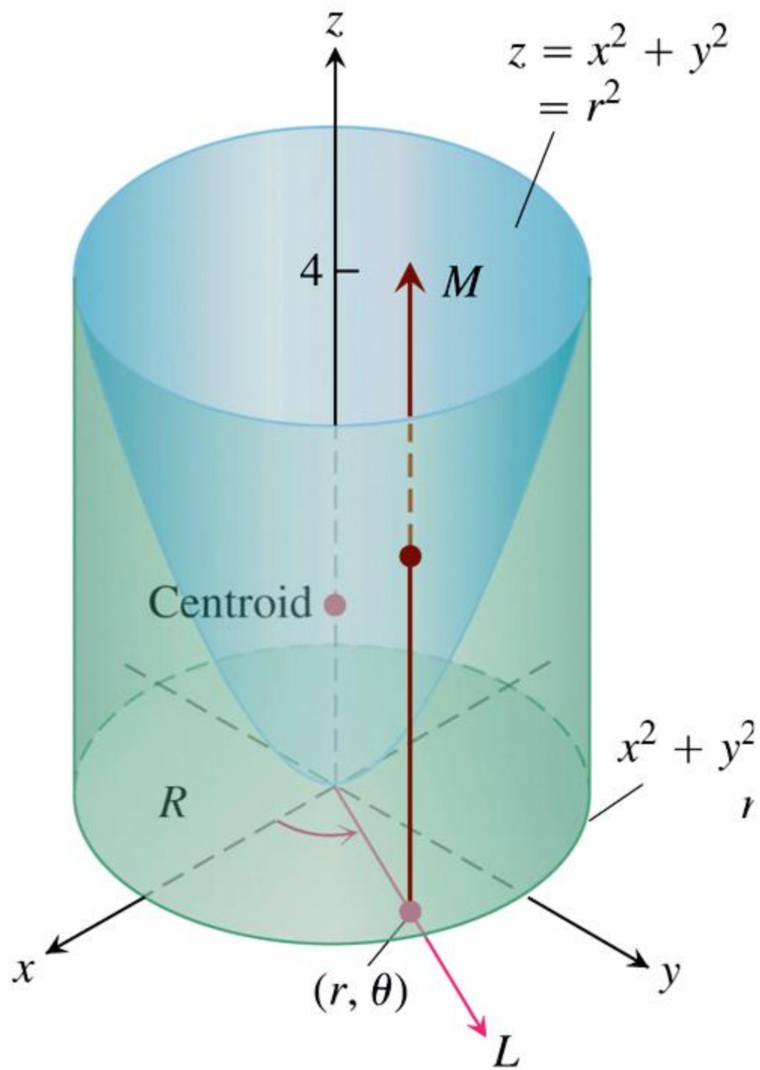


FIGURE 15.46 Example 2 shows how to find the centroid of this solid.

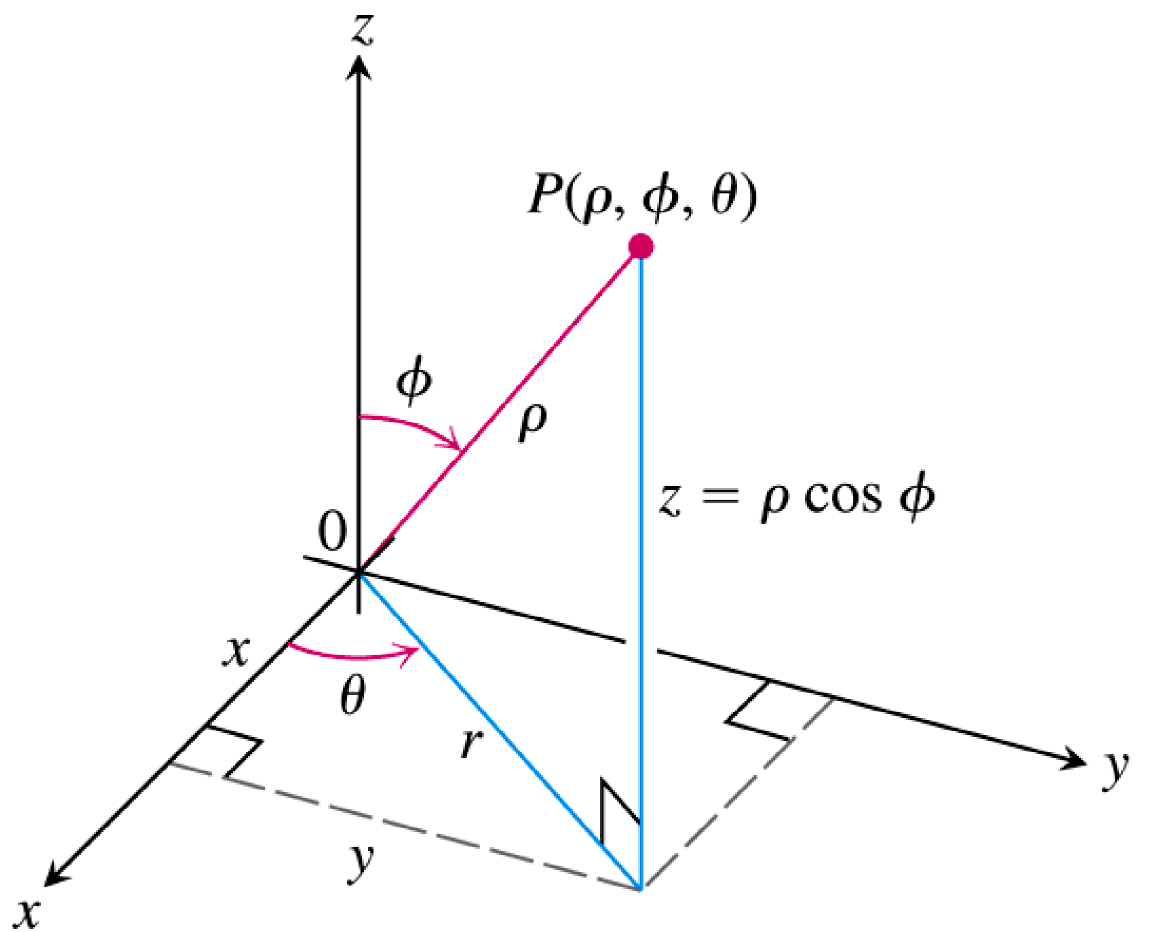


FIGURE 15.47 The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
3. θ is the angle from cylindrical coordinates $(0 \leq \theta \leq 2\pi)$.

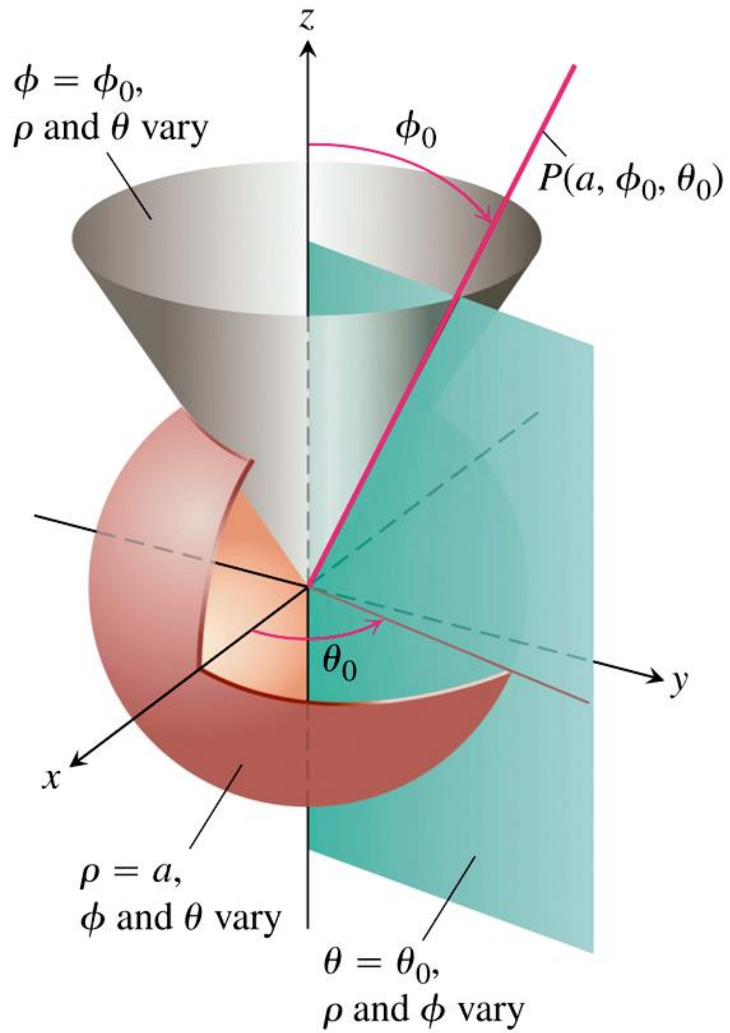


FIGURE 15.48 Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$\begin{aligned} r &= \rho \sin \phi, & x &= r \cos \theta = \rho \sin \phi \cos \theta, \\ z &= \rho \cos \phi, & y &= r \sin \theta = \rho \sin \phi \sin \theta, \\ \rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}. \end{aligned} \tag{1}$$

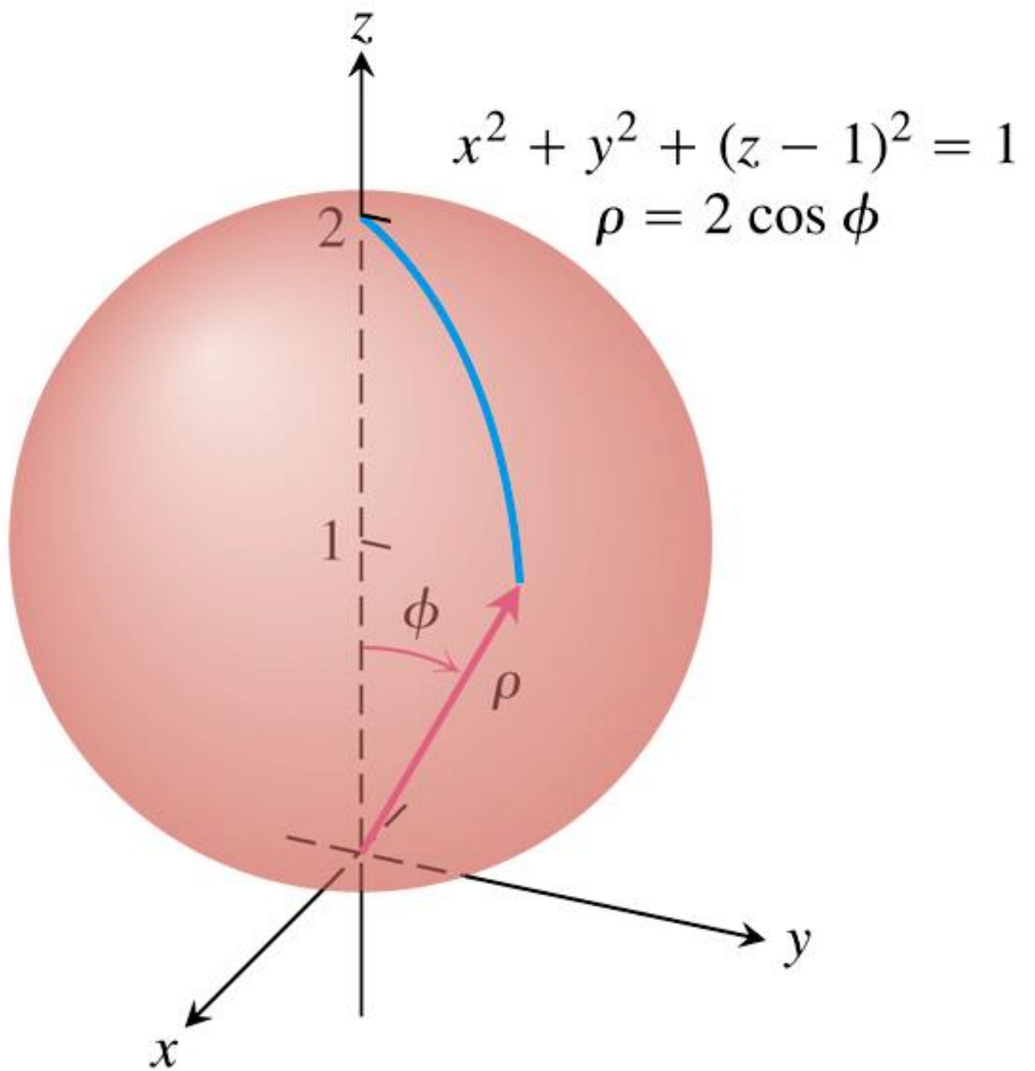


FIGURE 15.49 The sphere in Example 3.

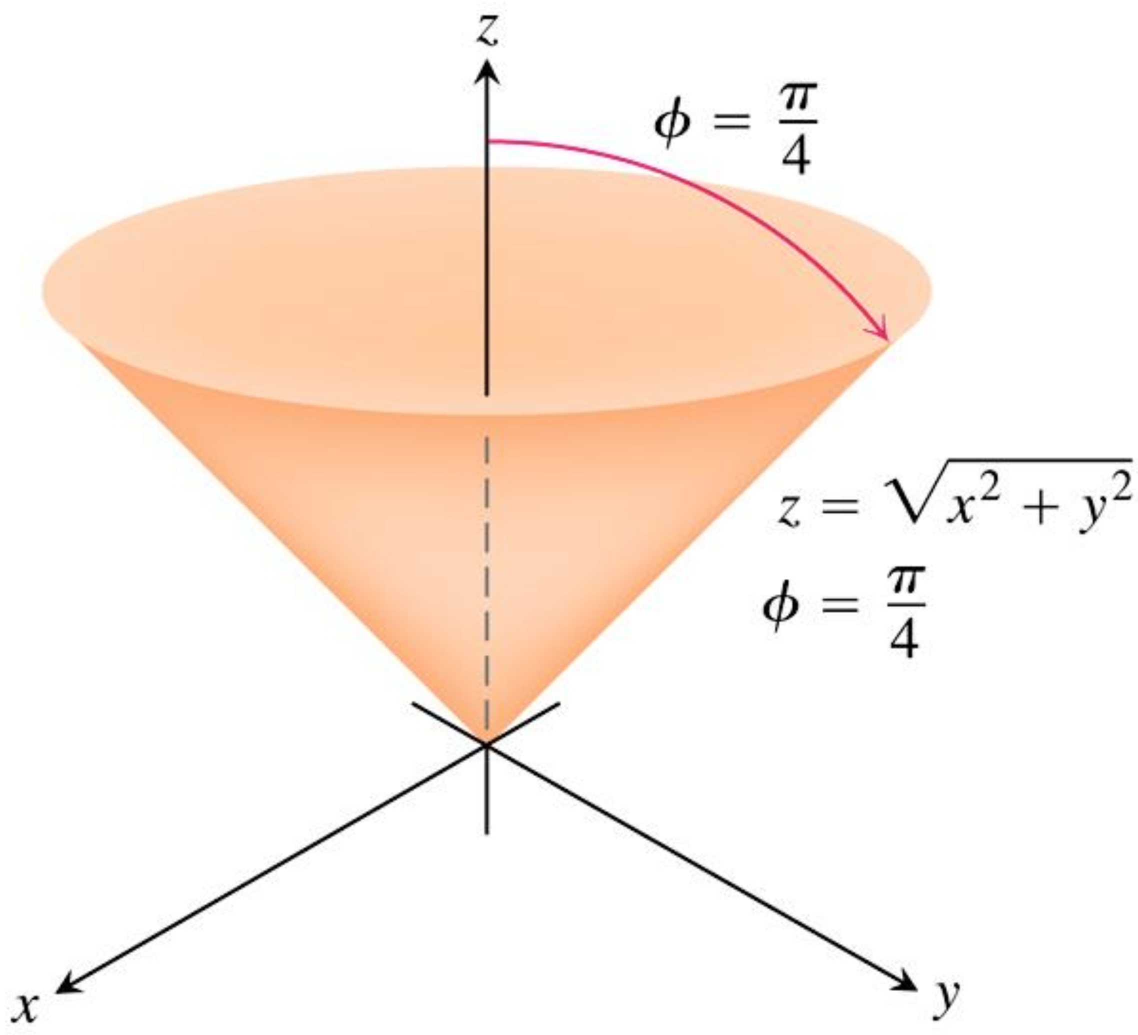


FIGURE 15.50 The cone in Example 4.

Volume Differential in Spherical Coordinates

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

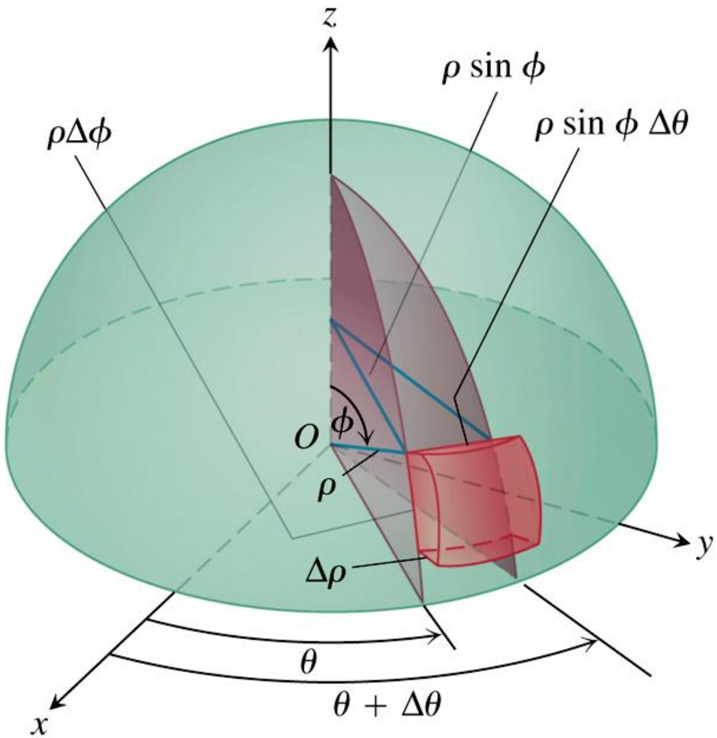


FIGURE 15.51 In spherical coordinates

$$\begin{aligned} dV &= d\rho \cdot \rho \, d\phi \cdot \rho \sin\phi \, d\theta \\ &= \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta. \end{aligned}$$

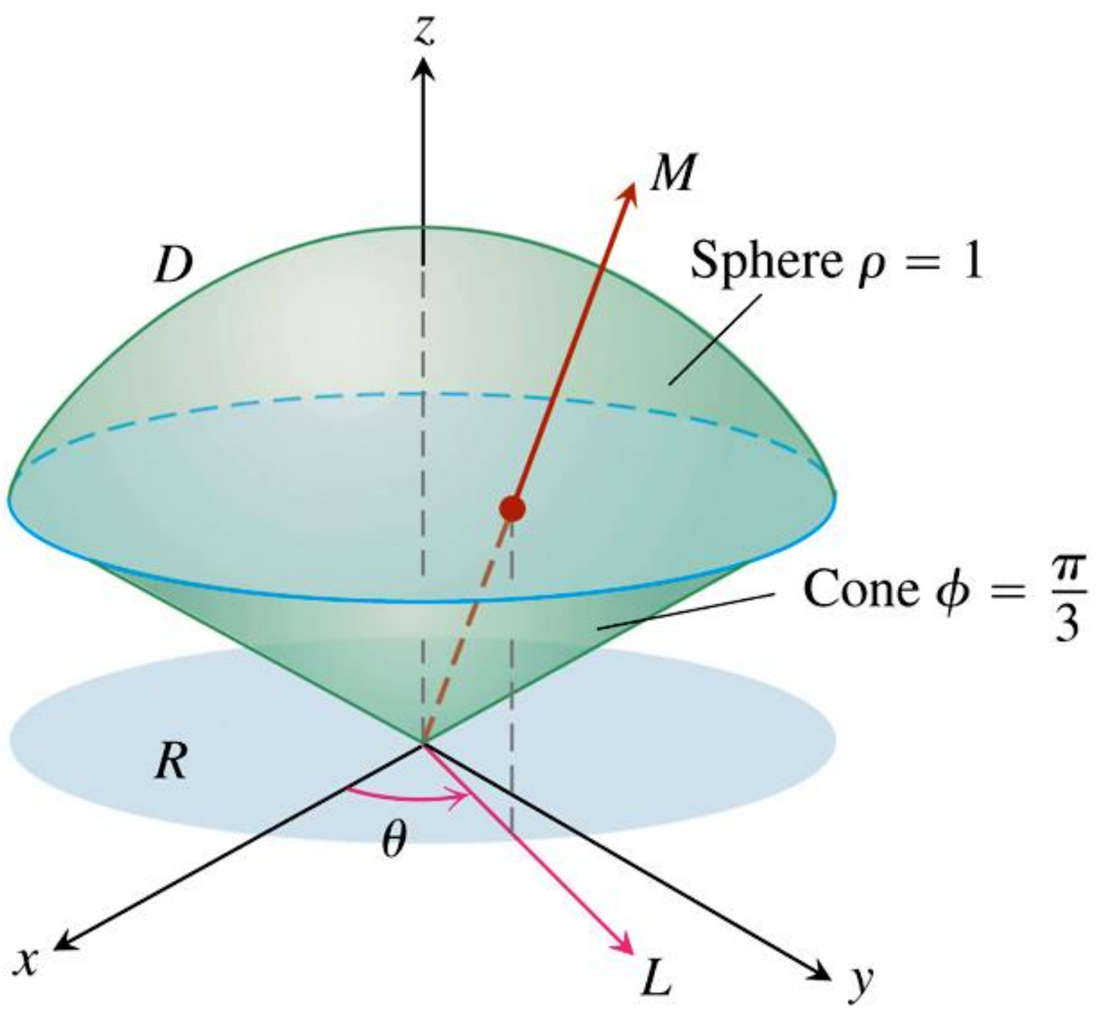


FIGURE 15.52 The ice cream cone in Example 5.

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

15.8

Substitutions in Multiple Integrals

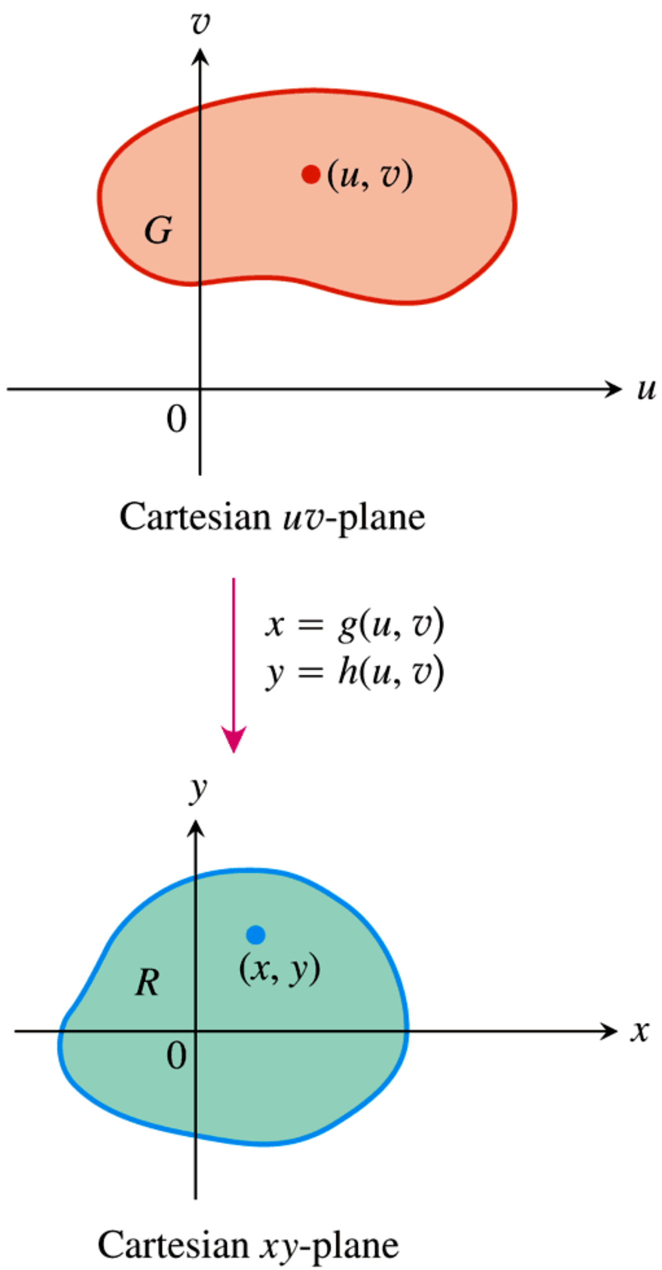


FIGURE 15.53 The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to change an integral over a region R in the xy-plane into an integral over a region G in the uv-plane by using Equation (1).

DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \qquad (2)$$

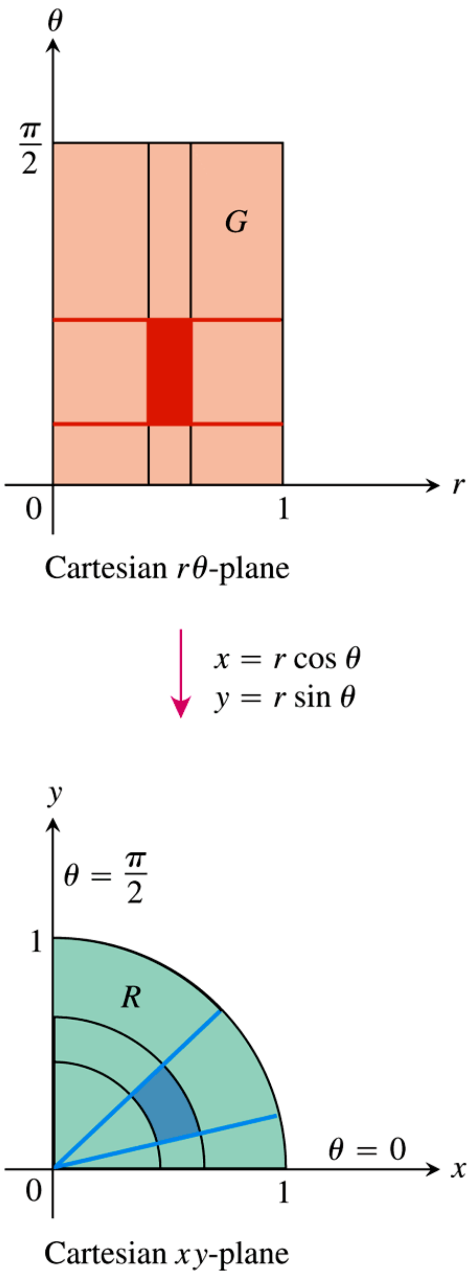


FIGURE 15.54 The equations $x = r\cos\theta$, $y = r\sin\theta$ transform G into R.

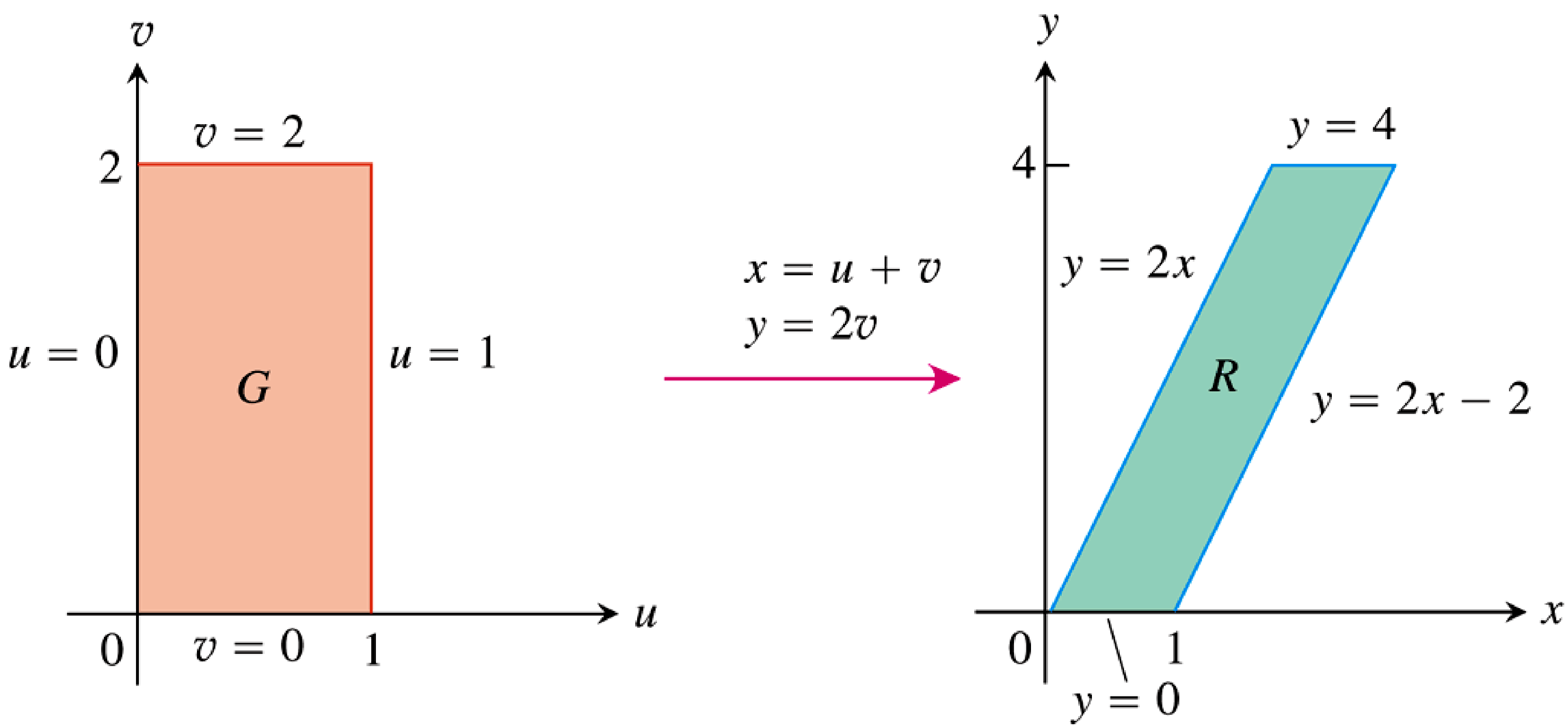


FIGURE 15.55 The equations $x = u + v$ and $y = 2v$ transform G into R. Reversing the transformation by the equations $u = (2x - y)/2$ and $v = y/2$ transforms R into G (Example 2).

xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

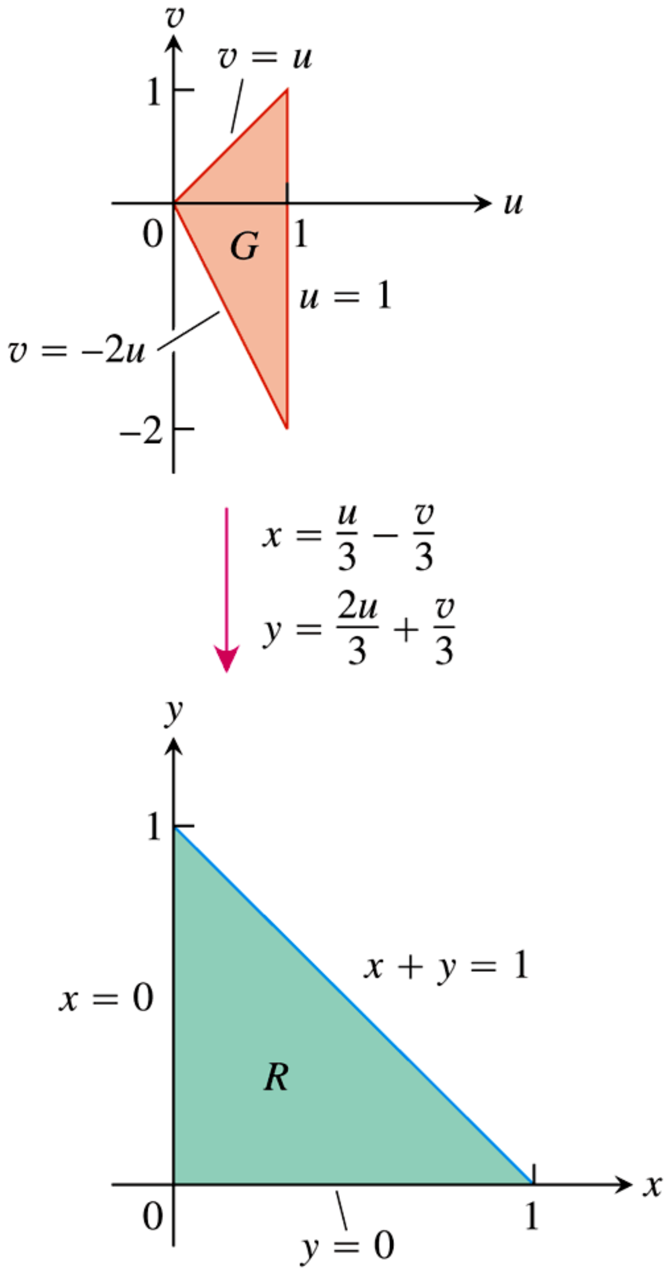


FIGURE 15.56 The equations $x = (u/3) - (v/3)$ and $y = (2u/3) + (v/3)$ transform G into R. Reversing the transformation by the equations $u = x + y$ and $v = y - 2x$ transforms R into G (Example 3).

xy-equations for the boundary of *R*	Corresponding *uv*-equations for the boundary of *G*	Simplified *uv*-equations
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

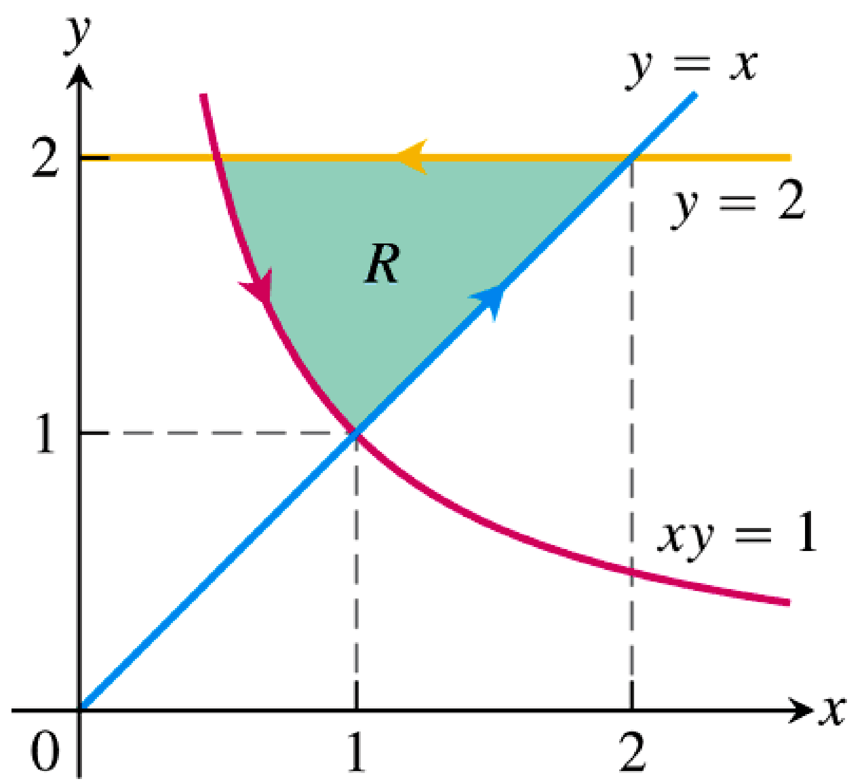


FIGURE 15.57 The region of integration R in Example 4.

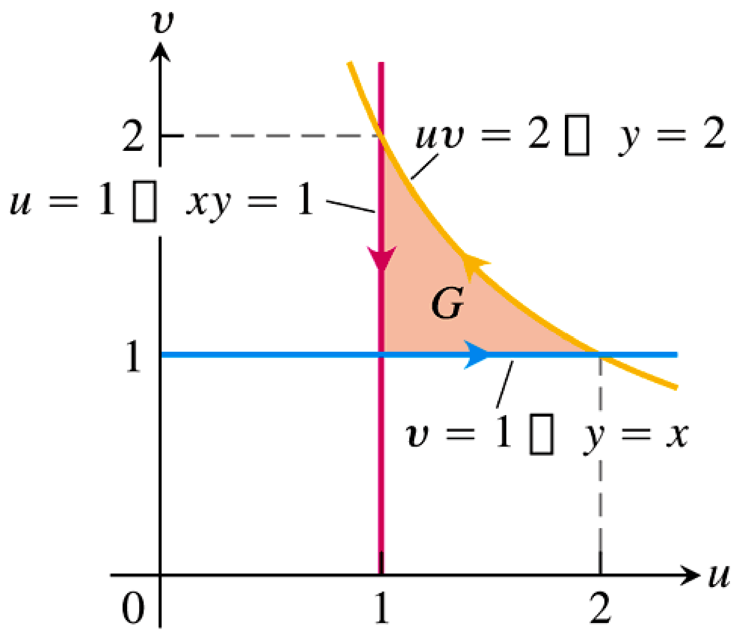


FIGURE 15.58 The boundaries of the region G correspond to those of region R in Figure 15.57. Notice as we move counterclockwise around the region R, we also move counterclockwise around the region G. The inverse transformation equations $u = \sqrt{xy}$, $v = \sqrt{y/x}$ produce the region G from the region R.

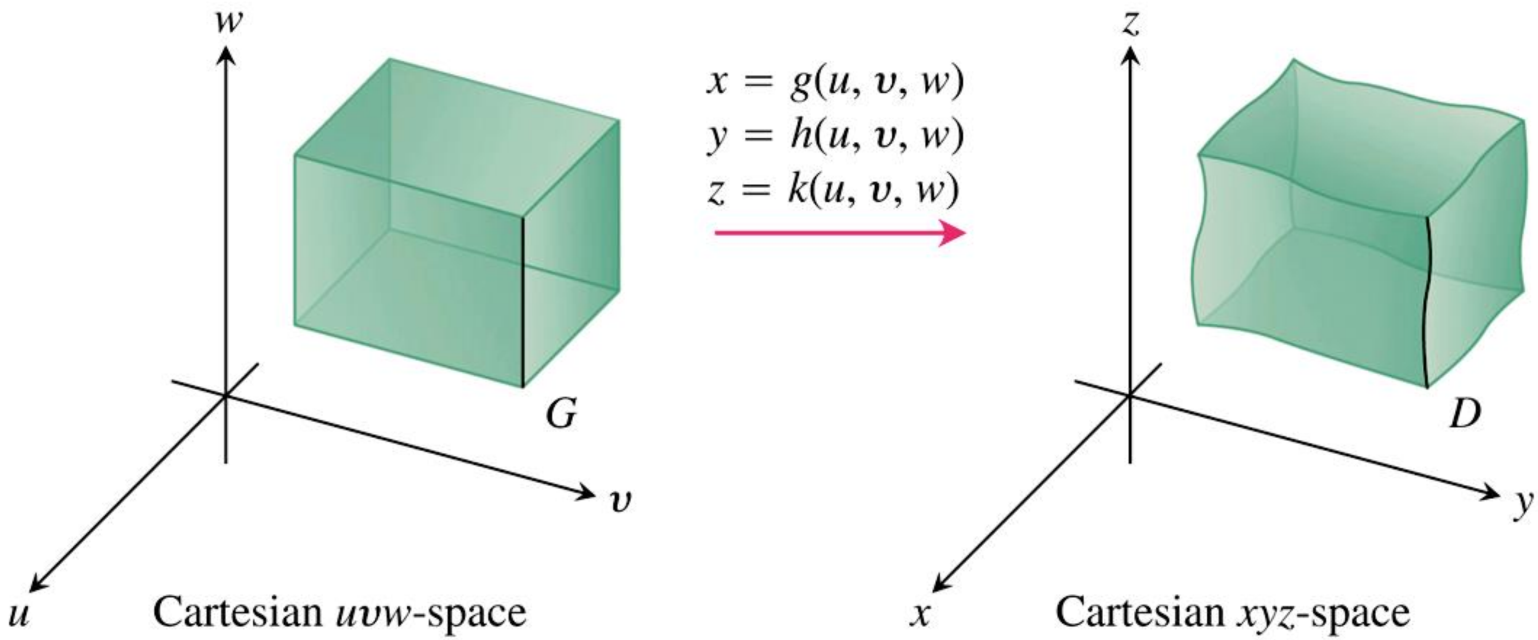


FIGURE 15.59 The equations $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$ allow us to change an integral over a region D in Cartesian xyz-space into an integral over a region G in Cartesian uyw-space using Equation (7).

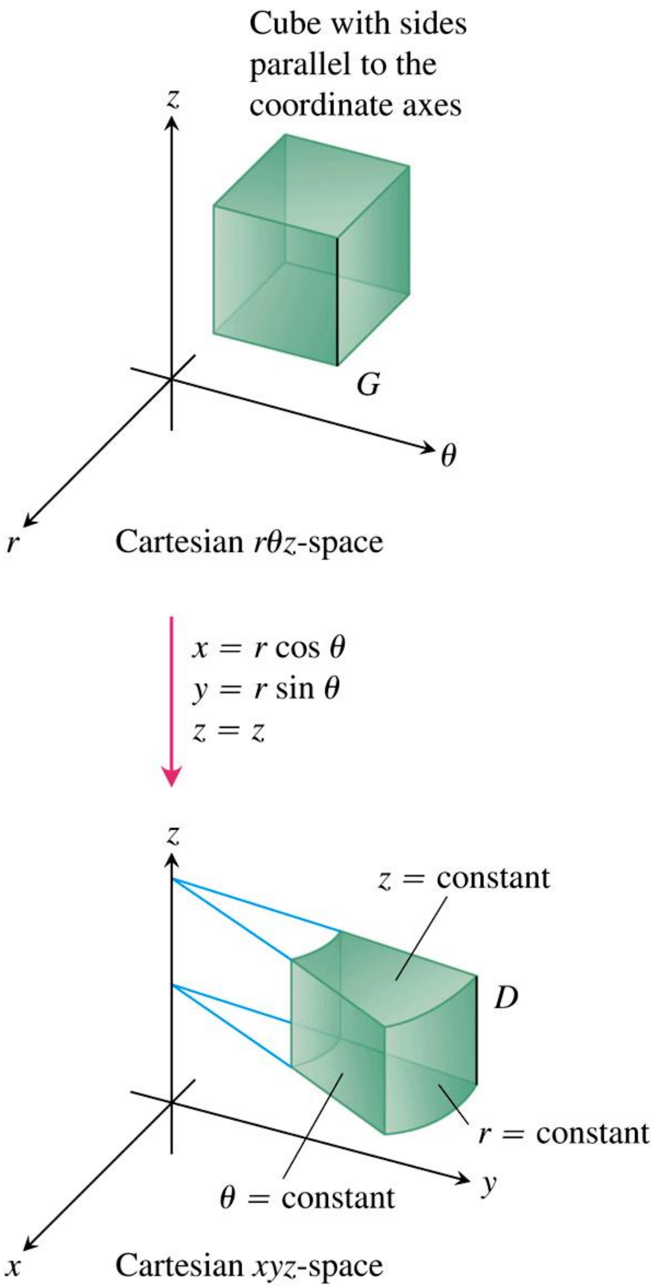


FIGURE 15.60 The equations $x = r \cos \theta, y = r \sin \theta$, and $z = z$ transform the cube G into a cylindrical wedge D.

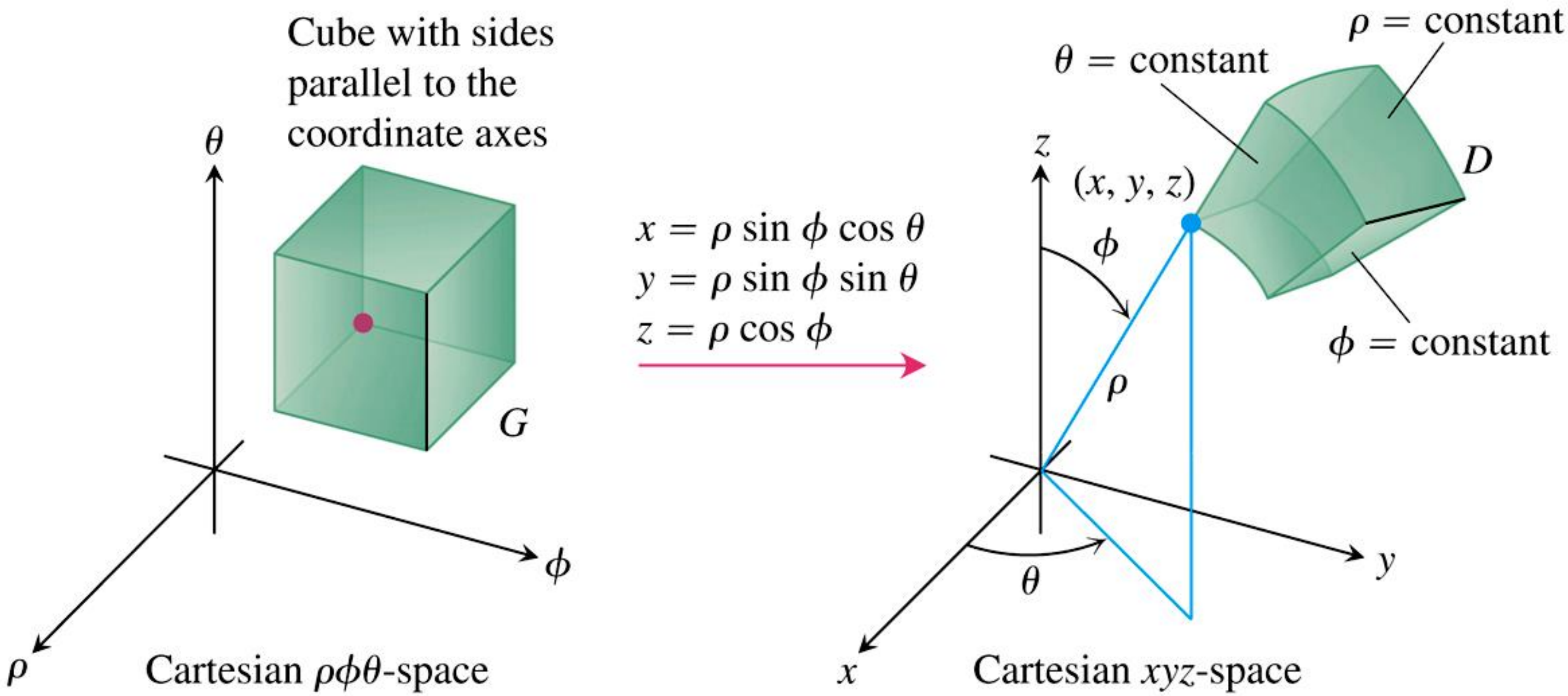


FIGURE 15.61 The equations $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$ transform the cube G into the spherical wedge D.

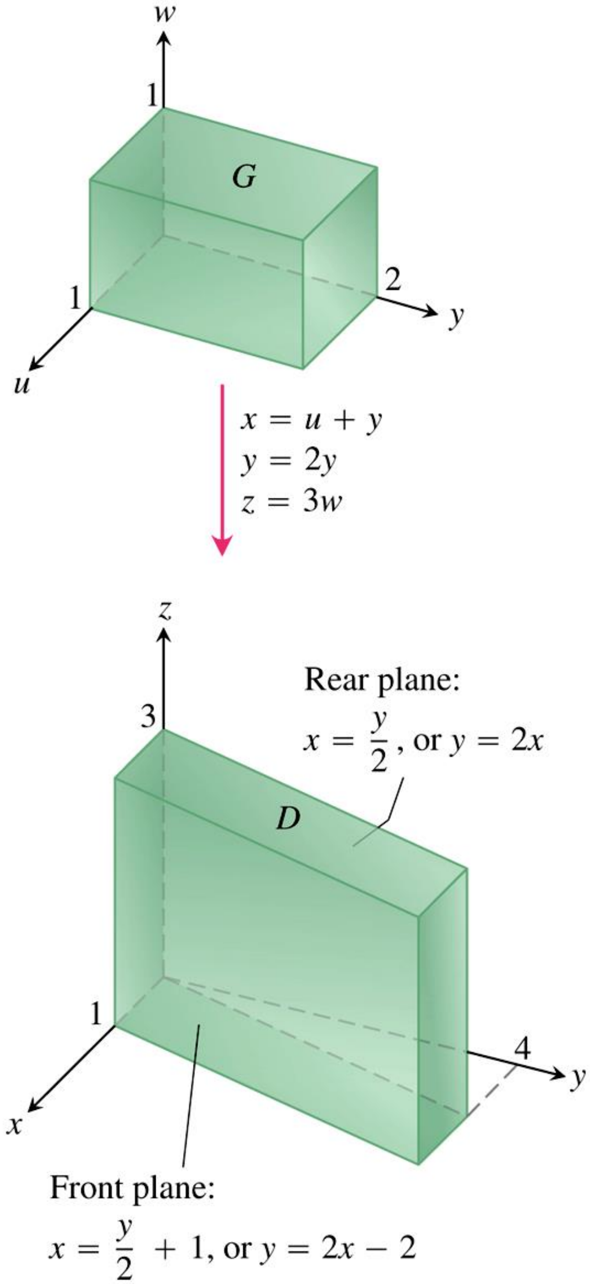


FIGURE 15.62 The equations $x = u + v$, $y = 2v$, and $z = 3w$ transform G into D. Reversing the transformation by the equations $u = (2x - y)/2$, $v = y/2$, and $w = z/3$ transforms D into G (Example 5).

xyz-equations for the boundary of *D*	Corresponding *uvw*-equations for the boundary of *G*	Simplified *uvw*-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$