Chapter 11

Parametric Equations and Polar Coordinates

11.1

Parametrizations of Plane Curves

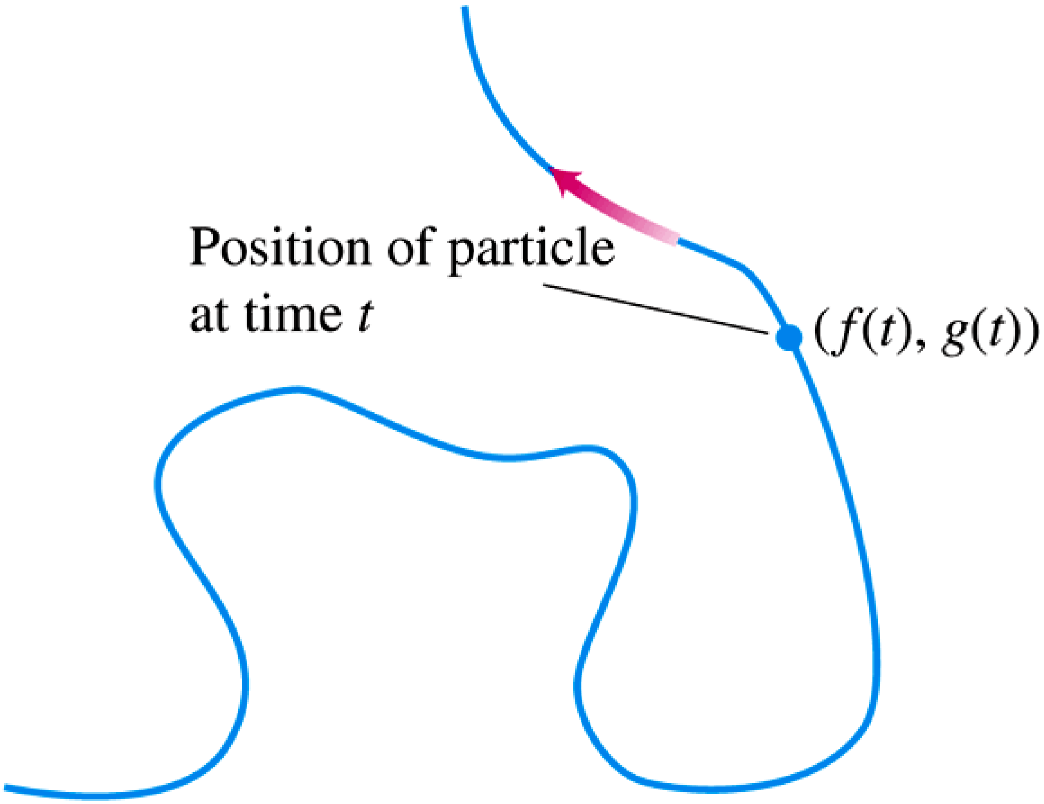


FIGURE 11.1 The curve or path traced by a particle moving in the xy-plane is not always the graph of a function or single equation.

DEFINITION If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

TABLE 11.1 Values of $x = t^2$ and $y = t + 1$ for selected values of t.

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

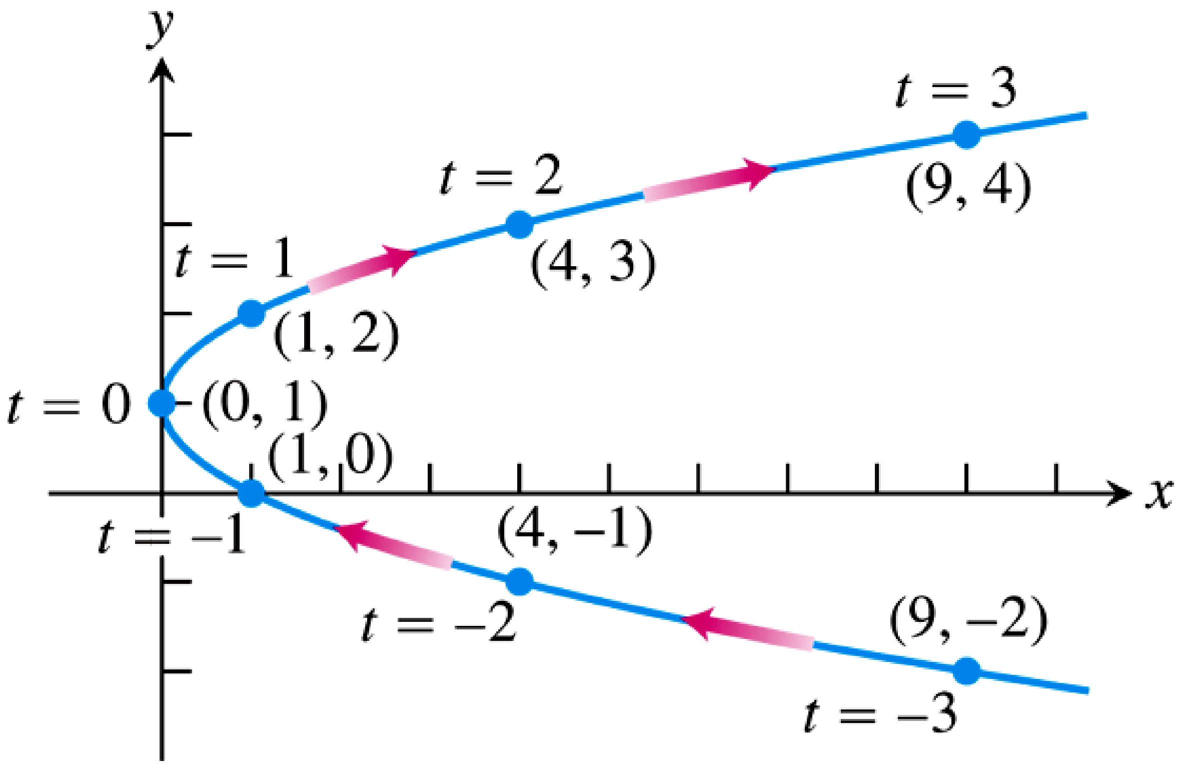


FIGURE 11.2 The curve given by the parametric equations $x = t^2$ and $y = t + 1$ (Example 1).

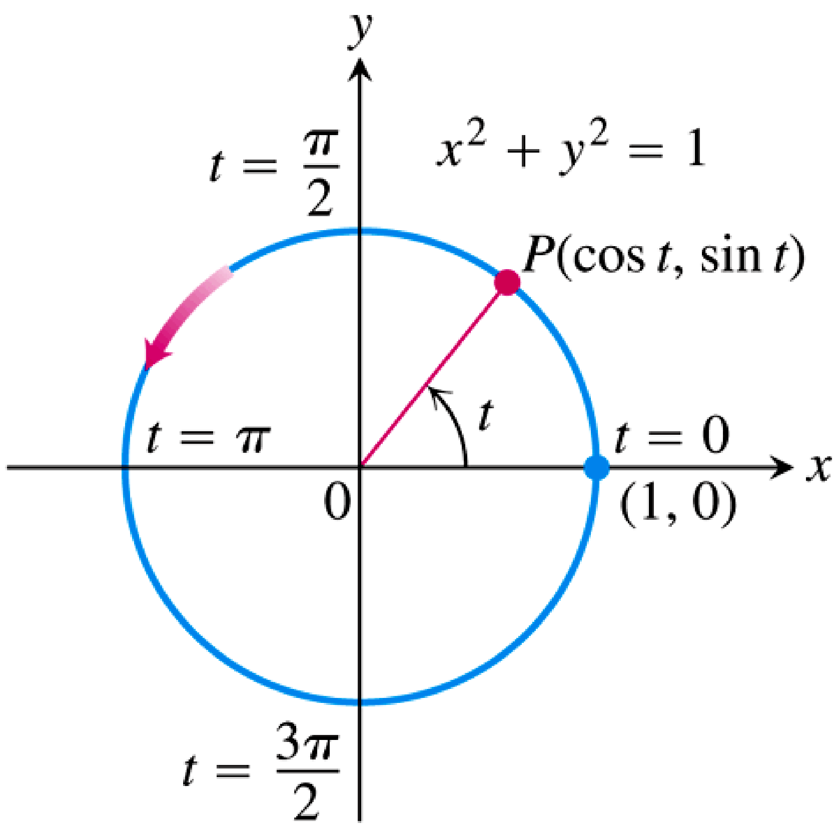


FIGURE 11.3 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 3).

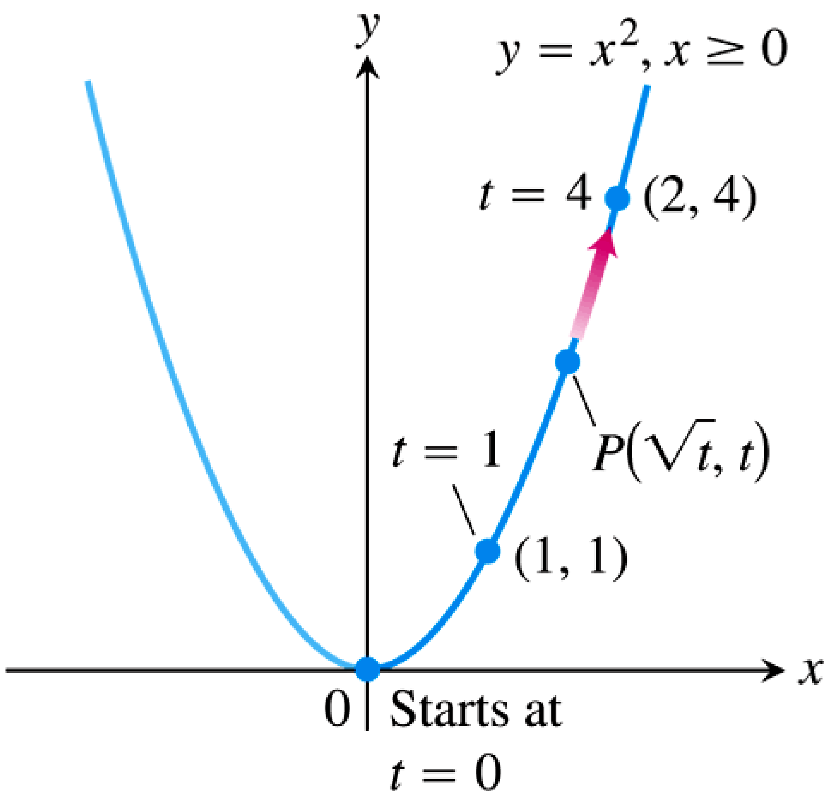


FIGURE 11.4 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the path of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 4).

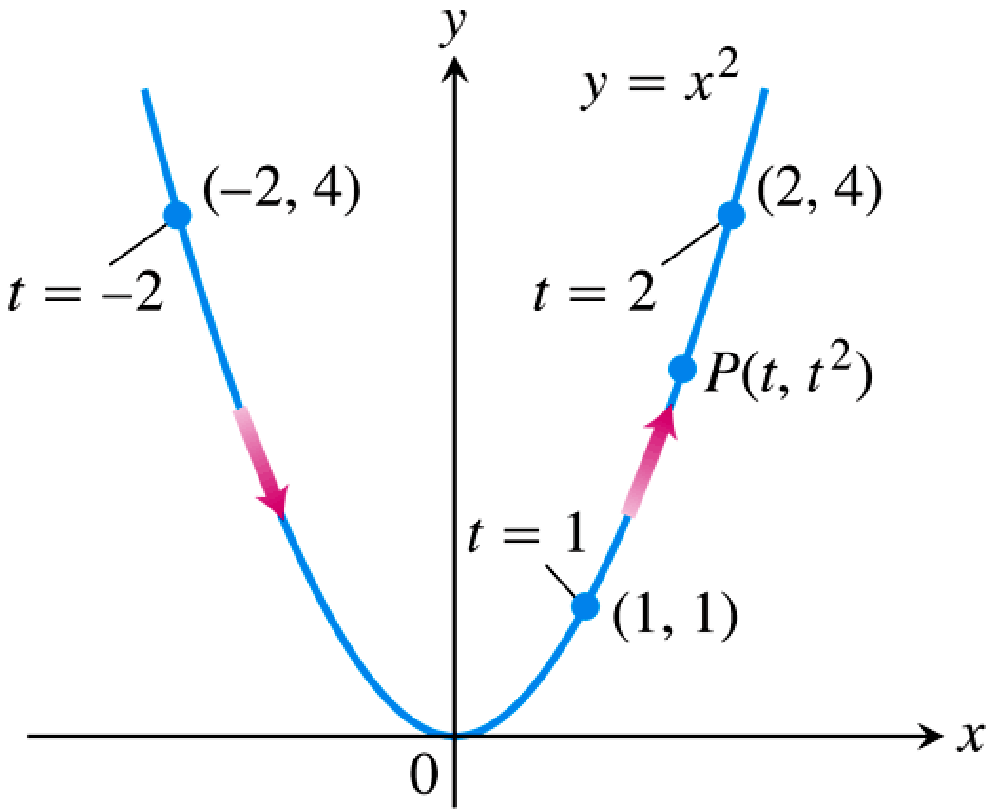


FIGURE 11.5 The path defined by $x = t, y = t^2, -\infty < t < \infty$ is the entire parabola $y = x^2$ (Example 5).

TABLE 11.2 Values of $x = t + (1/t)$ and $y = t - (1/t)$ for selected values of t.

t	$1/t$	x	y
0.1	10.0	10.1	−9.9
0.2	5.0	5.2	−4.8
0.4	2.5	2.9	−2.1
1.0	1.0	2.0	0.0
2.0	0.5	2.5	1.5
5.0	0.2	5.2	4.8
10.0	0.1	10.1	9.9

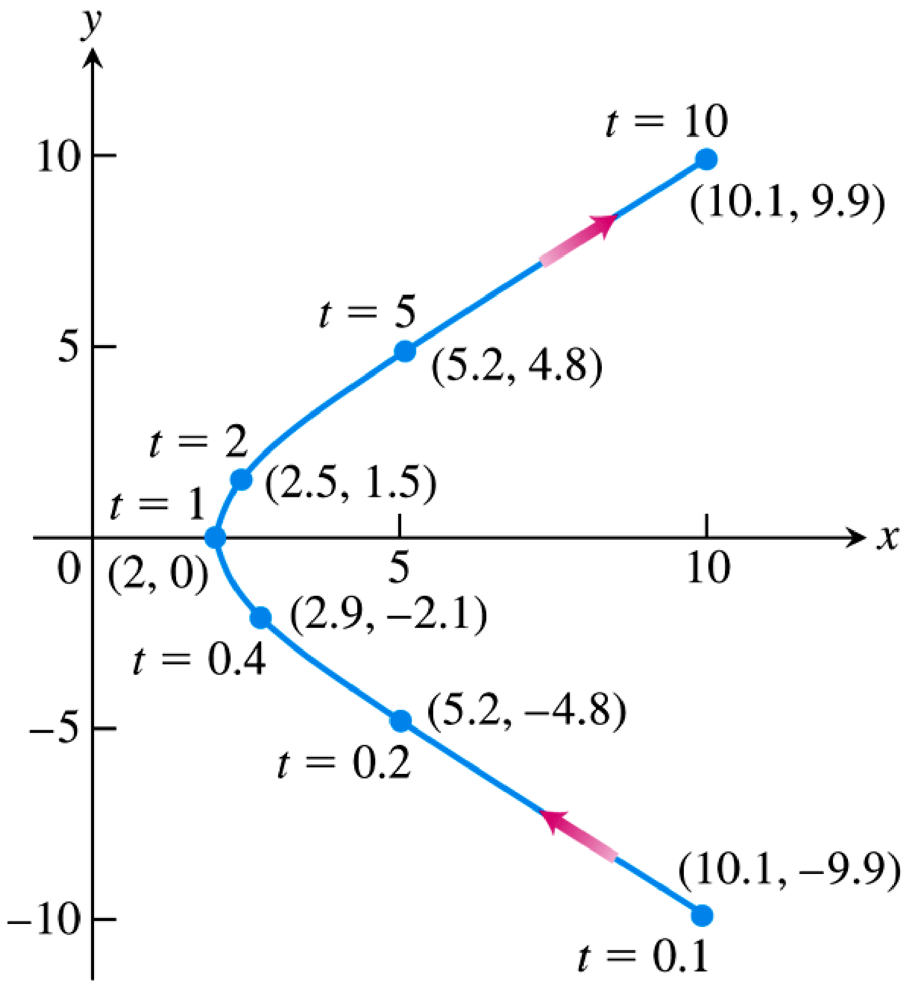


FIGURE 11.6 The curve for $x = t + (1/t)$, $y = t - (1/t)$, $t > 0$ in Example 7. (The part shown is for $0.1 \leq t \leq 10$.)

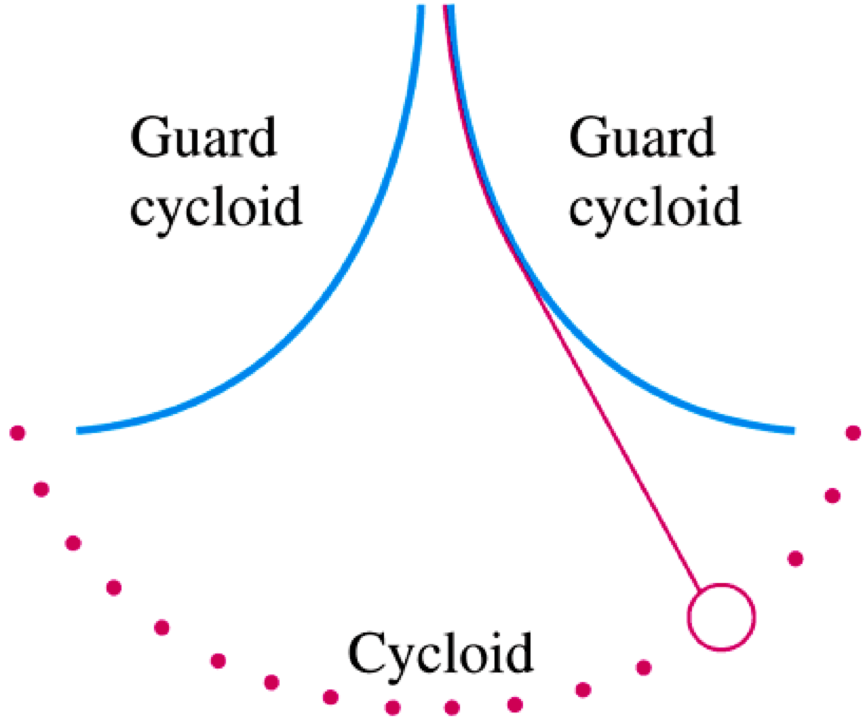


FIGURE 11.7 In Huygens' pendulum clock, the bob swings in a cycloid, so the frequency is independent of the amplitude.

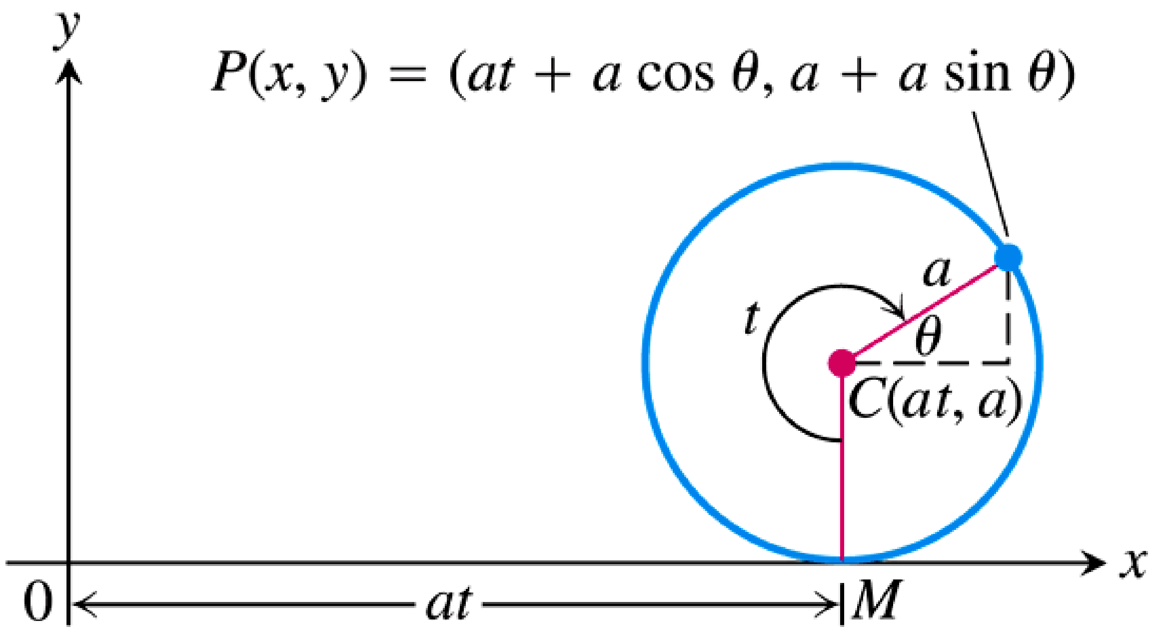


FIGURE 11.8 The position of $P(x, y)$ on the rolling wheel at angle t (Example 8).

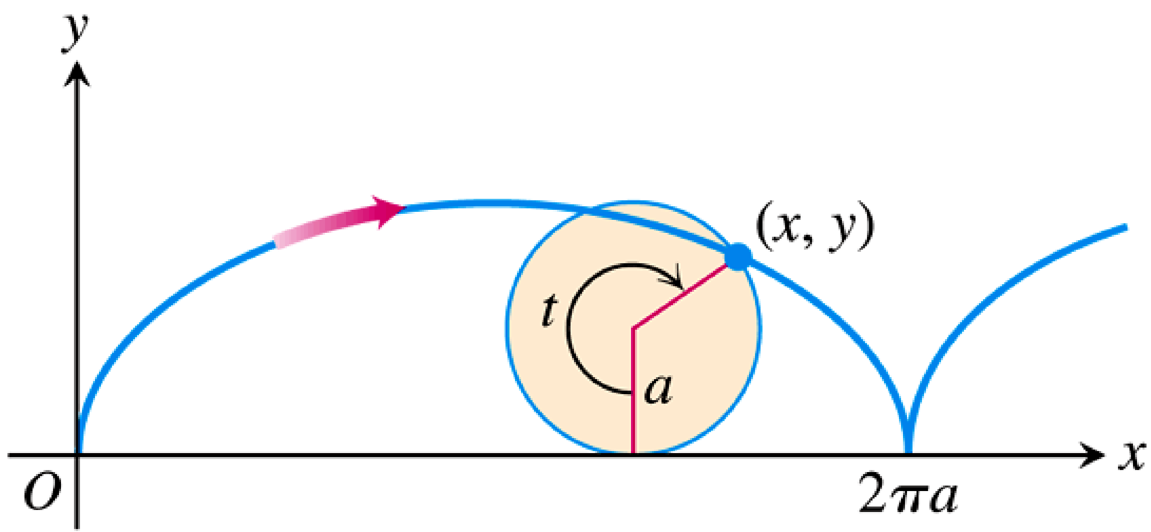


FIGURE 11.9 The cycloid curve $x = a(t - \sin t), y = a(1 - \cos t)$, for $t \geq 0$.

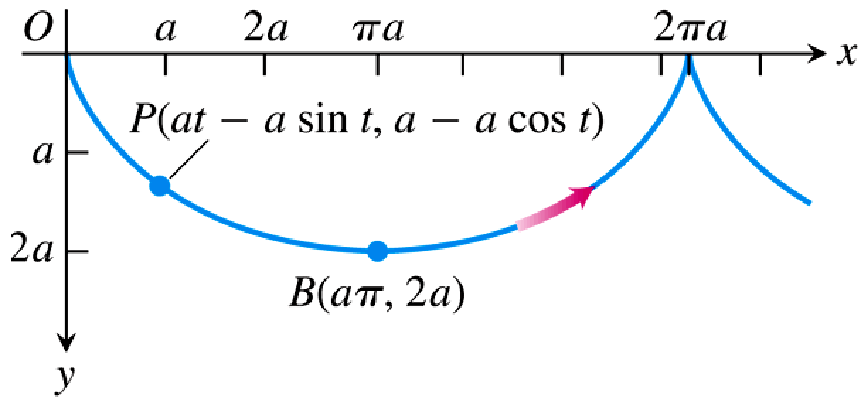


FIGURE 11.10 To study motion along an upside-down cycloid under the influence of gravity, we turn Figure 11.9 upside down. This points the y-axis in the direction of the gravitational force and makes the downward y-coordinates positive. The equations and parameter interval for the cycloid are still

$$\begin{aligned} x &= a(t - \sin t), \\ y &= a(1 - \cos t), \quad t \geq 0. \end{aligned}$$

The arrow shows the direction of increasing t.

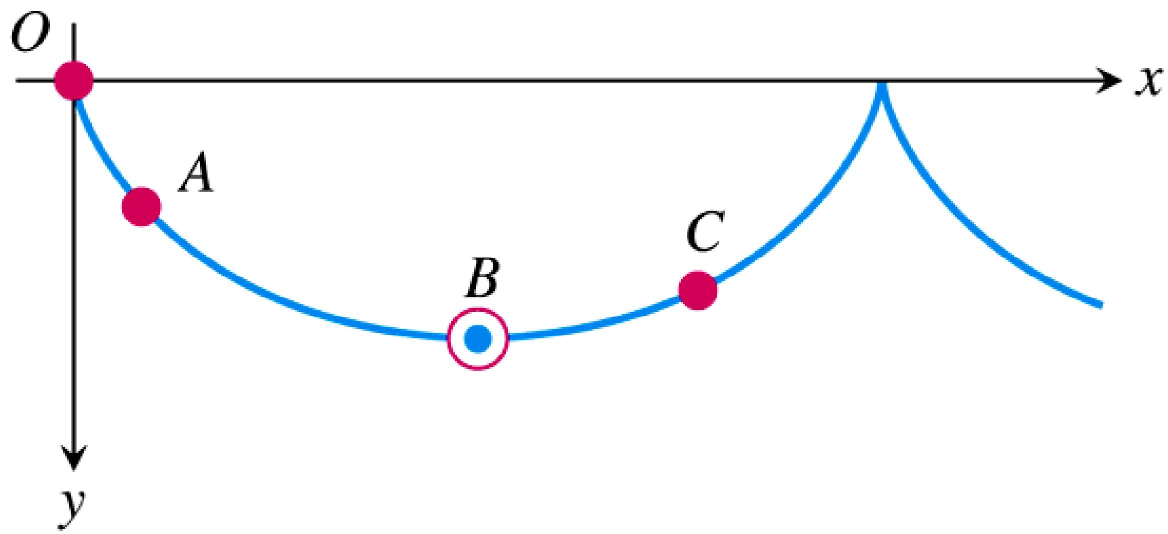


FIGURE 11.11 Beads released simultaneously on the upside-down cycloid at O, A, and C will reach B at the same time.

11.2

Calculus with Parametric Curves

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{1}$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{2}$$

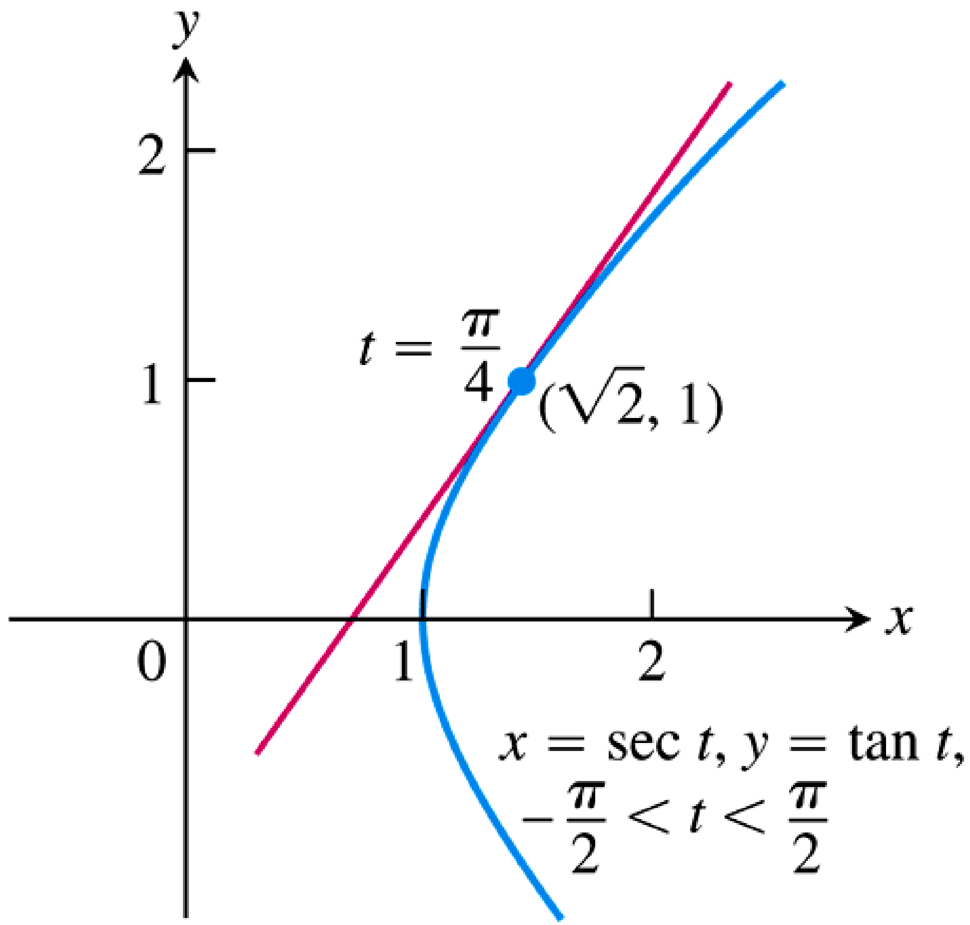


FIGURE 11.12 The curve in Example 1 is the right-hand branch of the hyperbola $x^2 - y^2 = 1$.

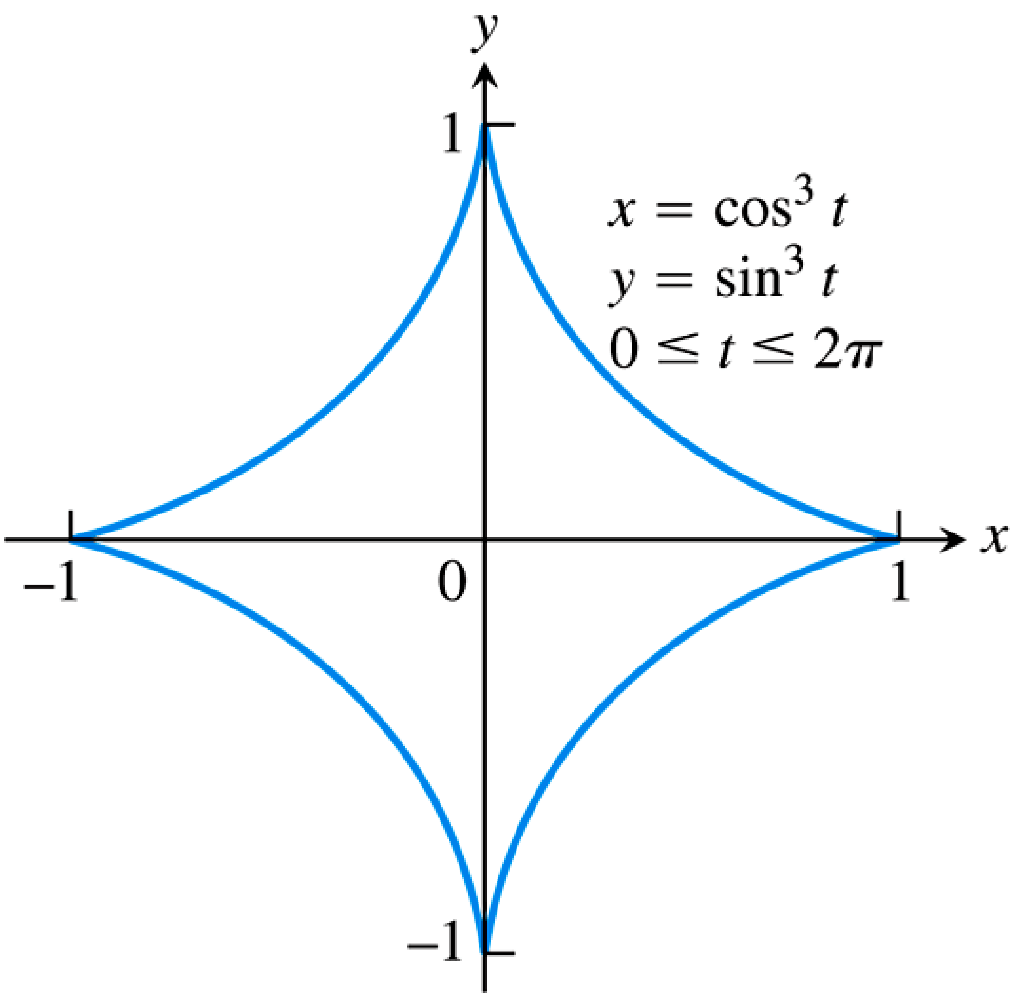


Figure 11.13 The astroid in Example 3.

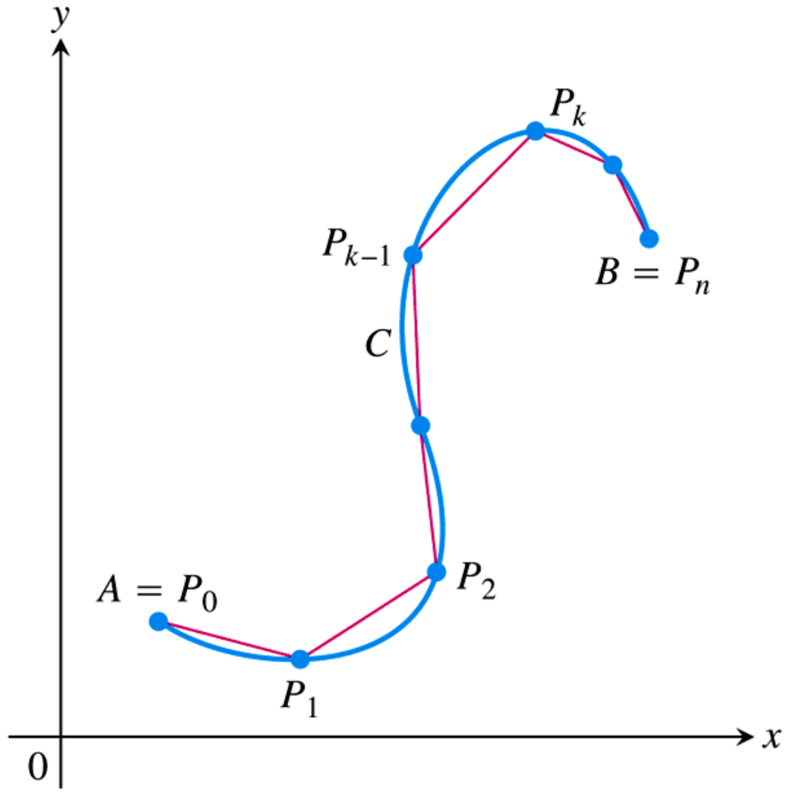


FIGURE 11.14 The smooth curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

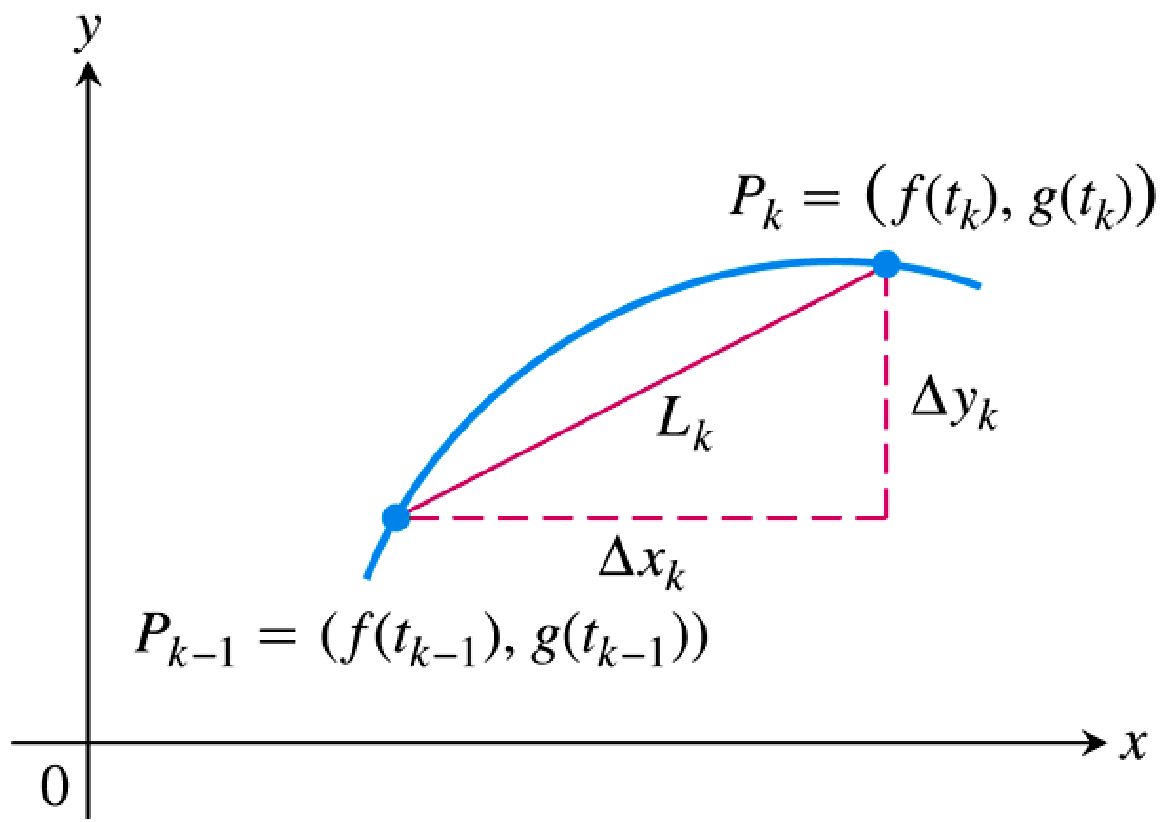


FIGURE 11.15 The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

DEFINITION If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

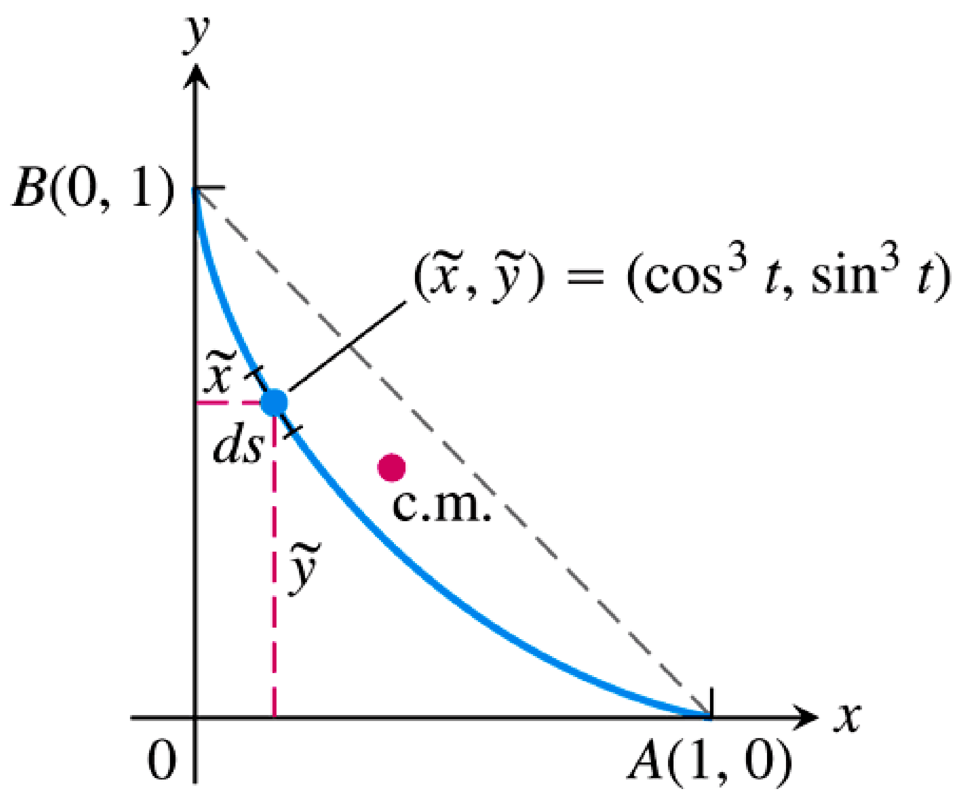


FIGURE 11.16 The centroid (c.m.) of the astroid arc in Example 6.

Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{5}$$

2. Revolution about the y-axis ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{6}$$

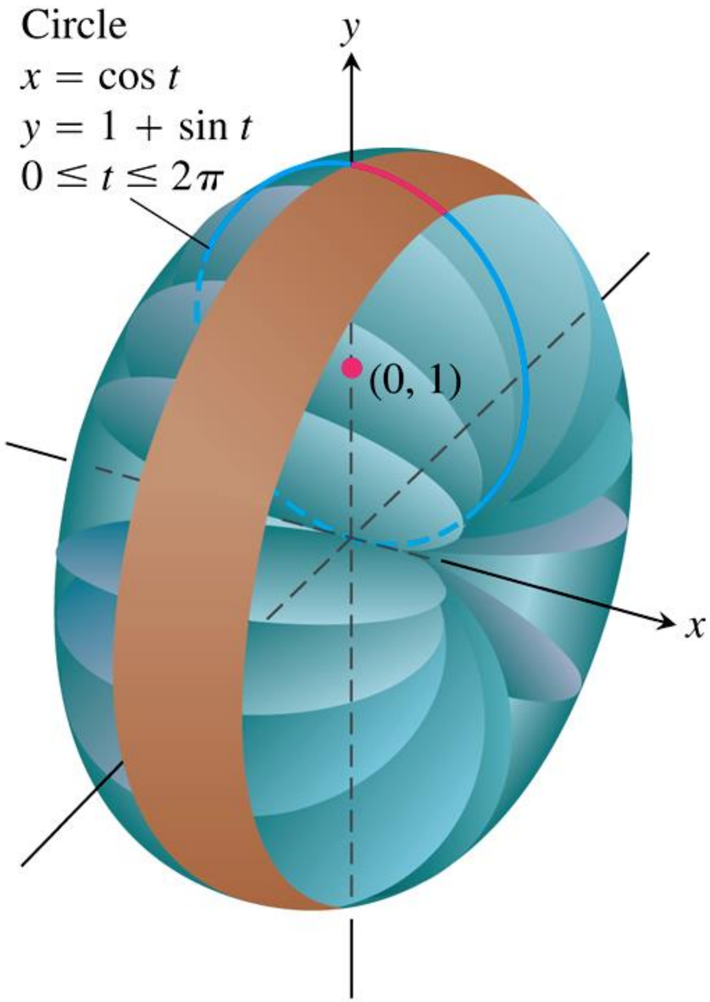


FIGURE 11.17 In Example 7 we calculate the area of the surface of revolution swept out by this parametrized curve.

11.3

Polar Coordinates

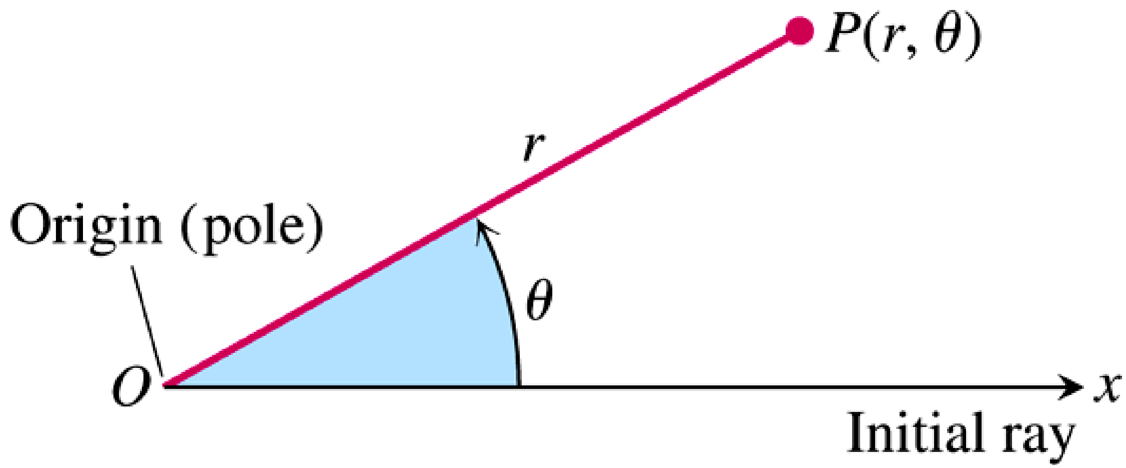


FIGURE 11.18 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

Polar Coordinates

$$P(r, \theta)$$

r: Directed distance from O to P

θ: Directed angle from initial ray to OP

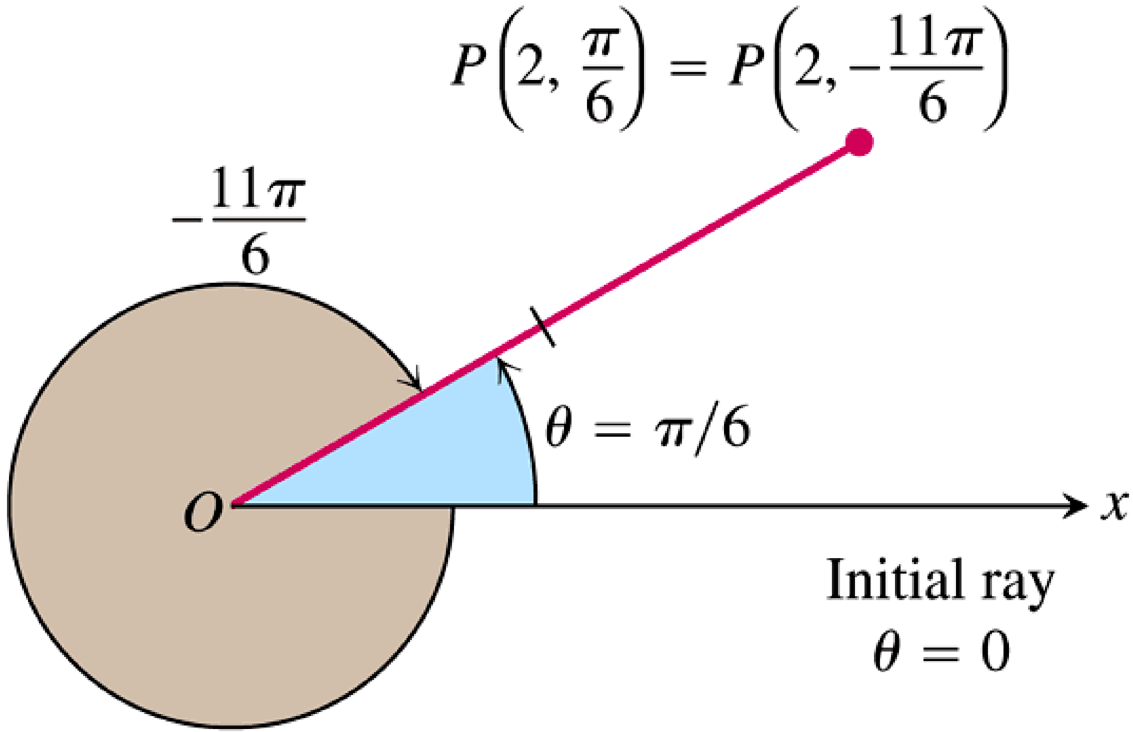


FIGURE 11.19 Polar coordinates are not unique.

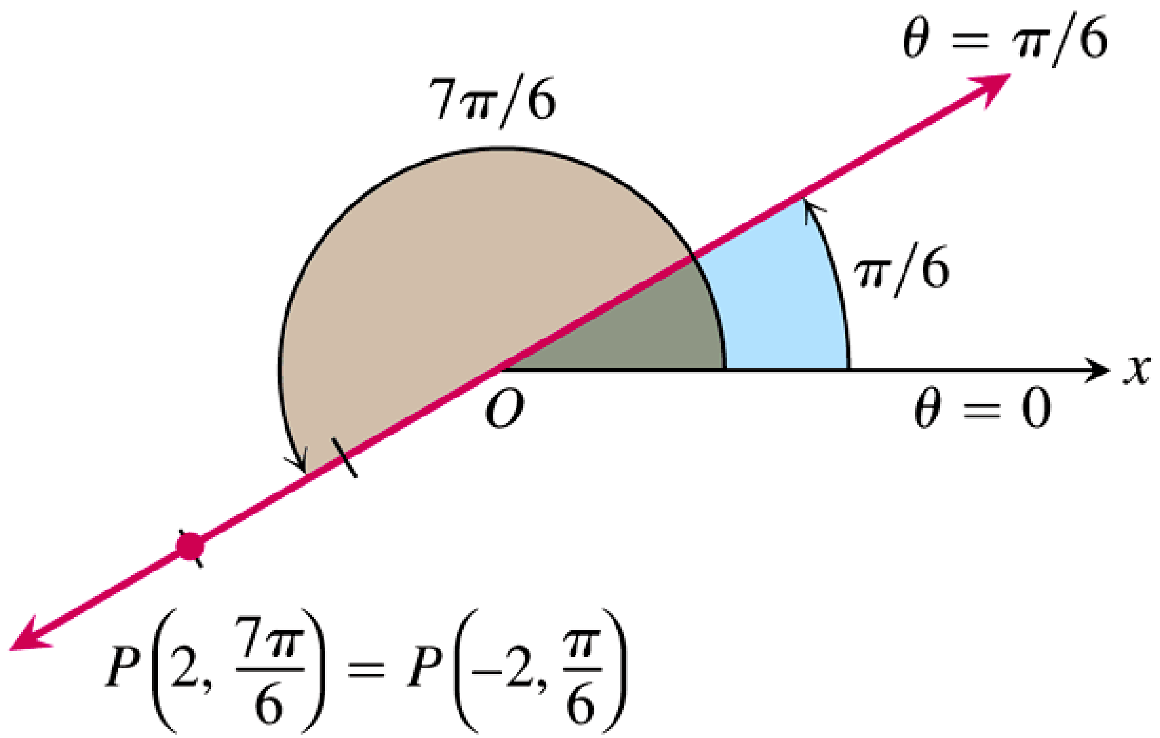


FIGURE 11.20 Polar coordinates can have negative r-values.

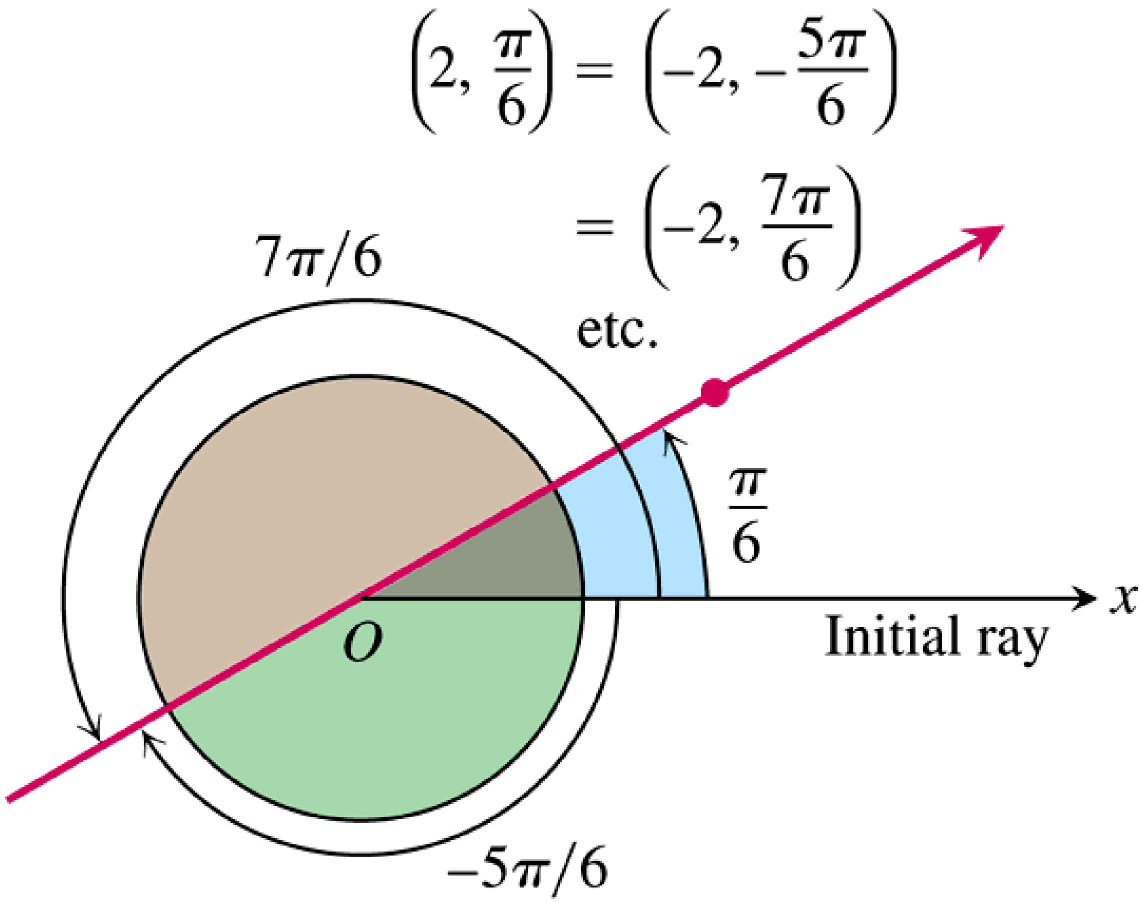


FIGURE 11.21 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

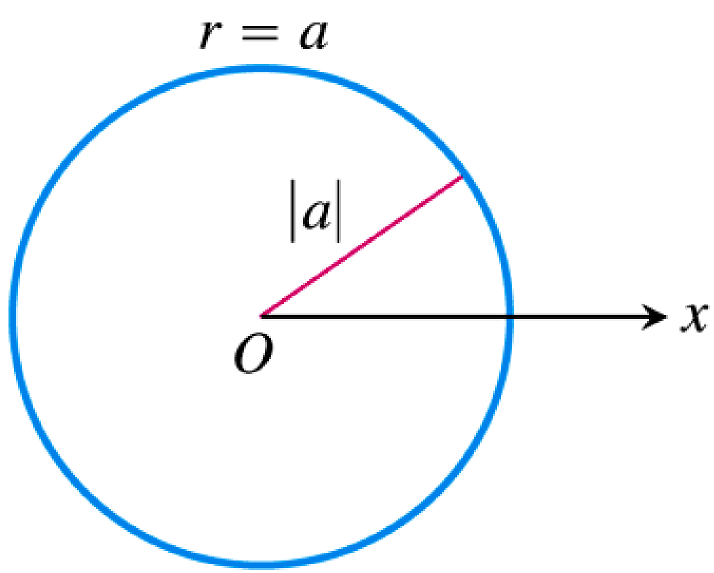


FIGURE 11.22 The polar equation for a circle is $r = a$.

Equation	Graph
$r = a$	Circle of radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

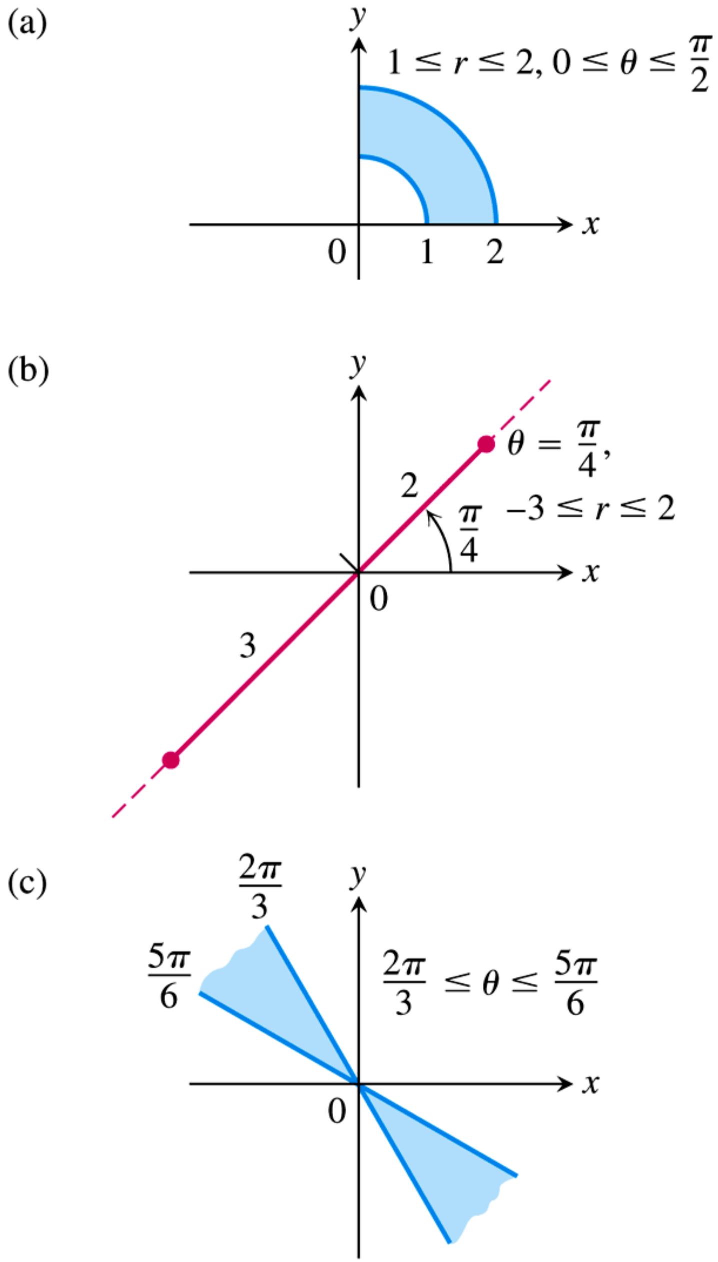


FIGURE 11.23 The graphs of typical inequalities in r and θ (Example 3).

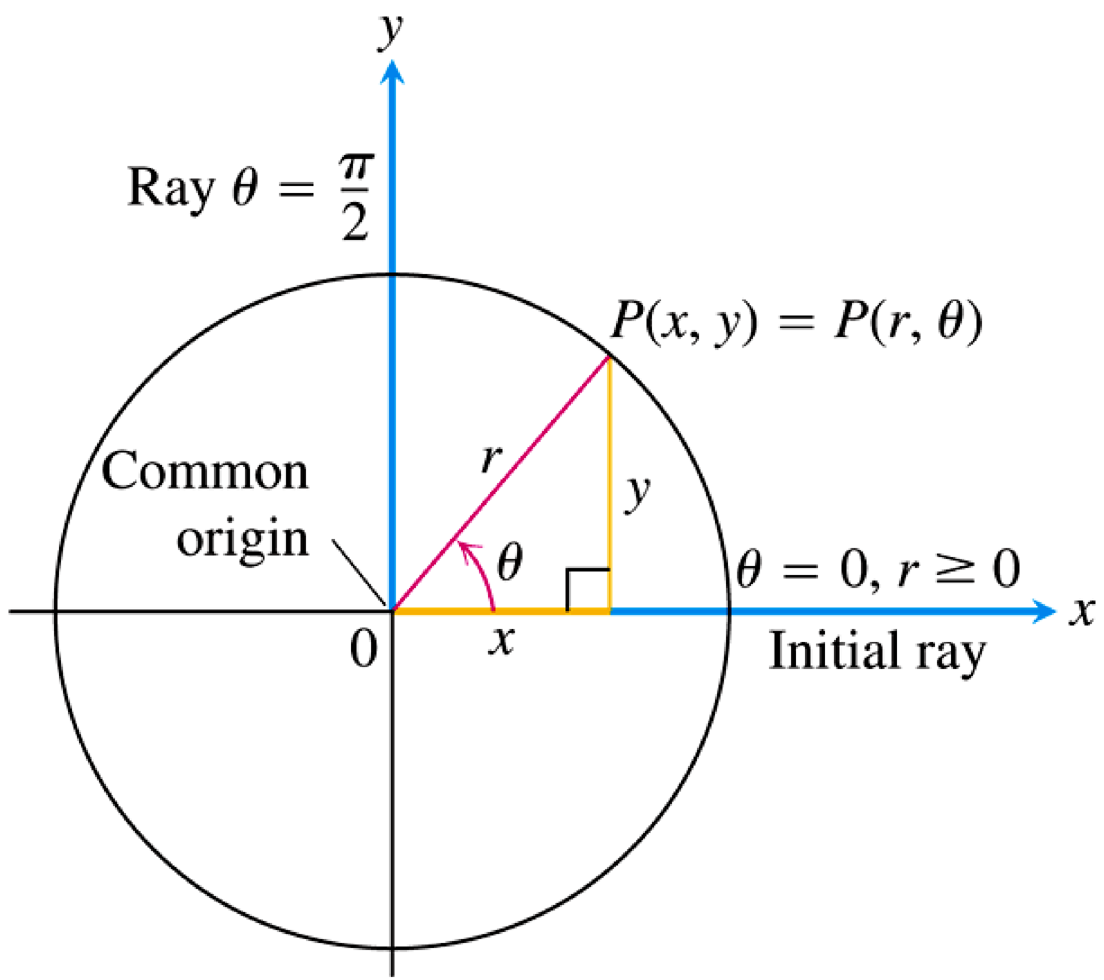


FIGURE 11.24 The usual way to relate polar and Cartesian coordinates.

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

EXAMPLE 4 Here are some equivalent equations expressed in terms of both polar coordinates and Cartesian coordinates.

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Some curves are more simply expressed with polar coordinates; others are not.

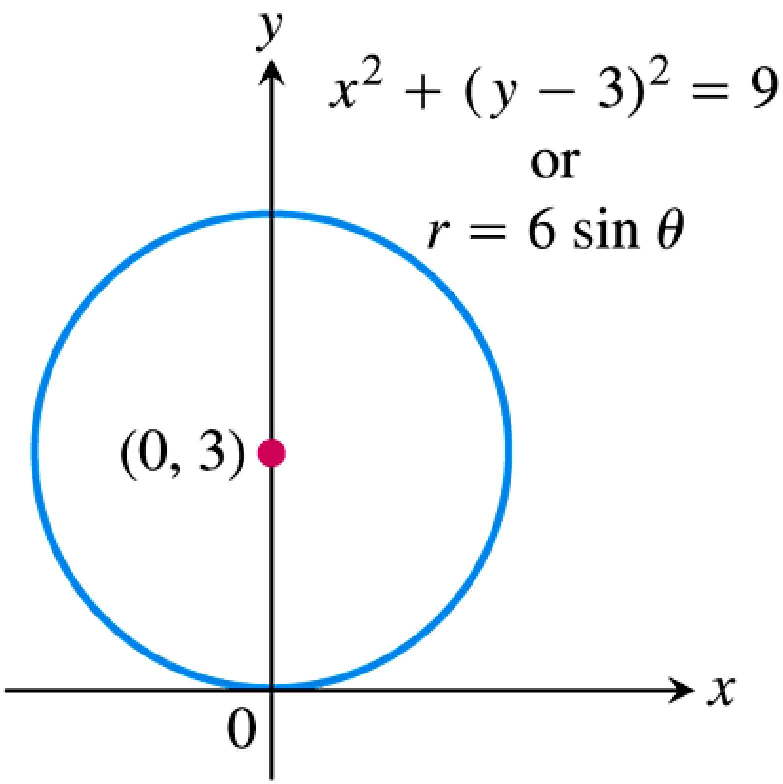


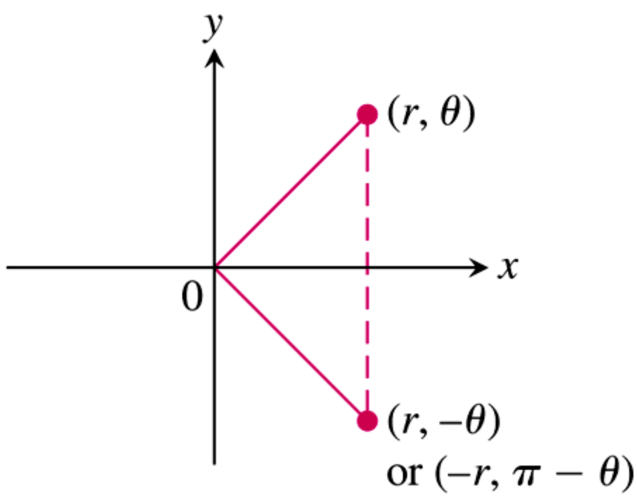
FIGURE 11.25 The circle in Example 5.

11.6

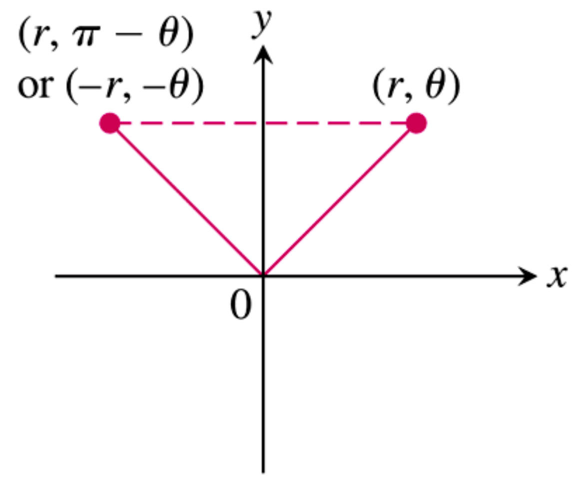
Graphing in Polar Coordinates

Symmetry Tests for Polar Graphs

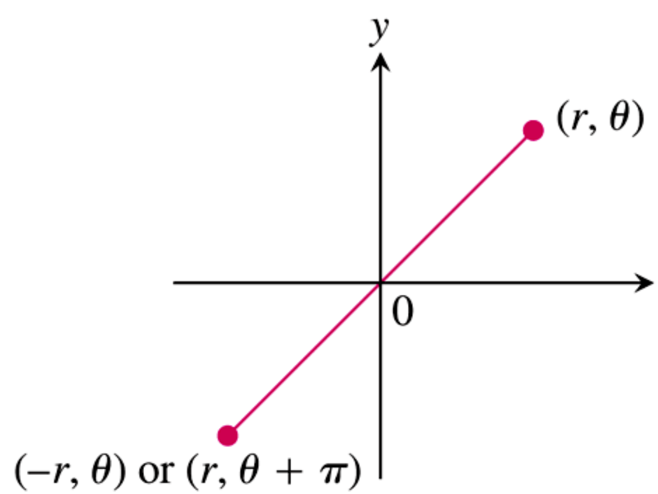
1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 11.26a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 11.26b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 11.26c).



(a) About the x-axis



(b) About the y-axis



(c) About the origin

FIGURE 11.26 Three tests for symmetry in polar coordinates.

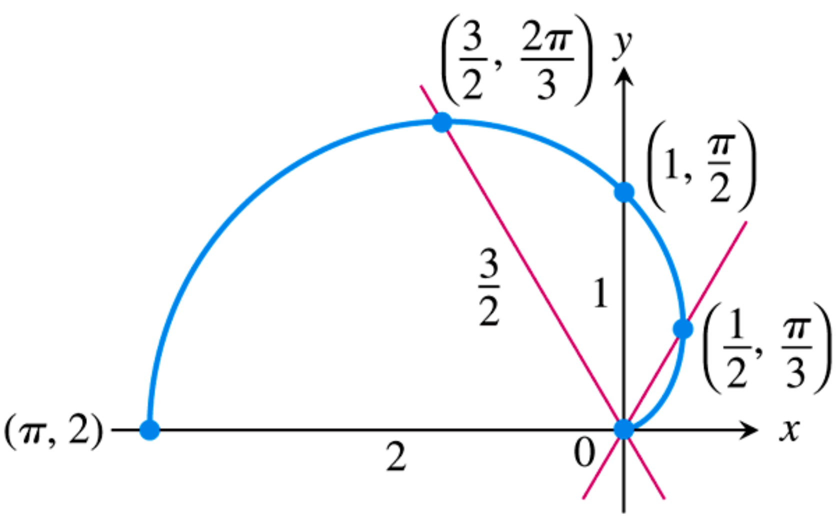
Slope of the Curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta},$$

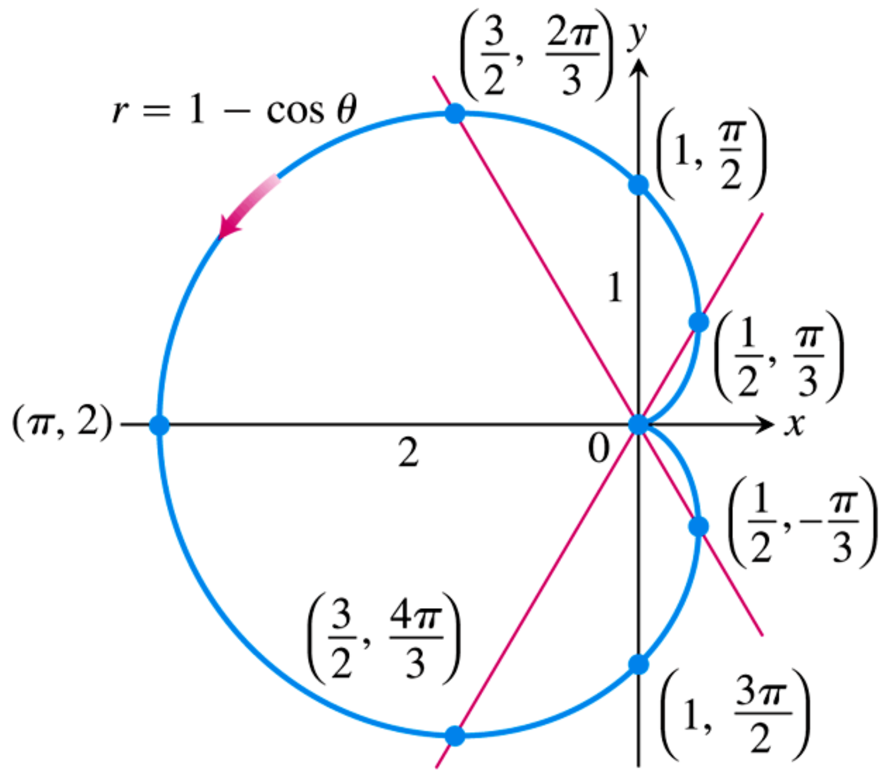
provided $dx/d\theta \neq 0$ at (r, θ).

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



(b)

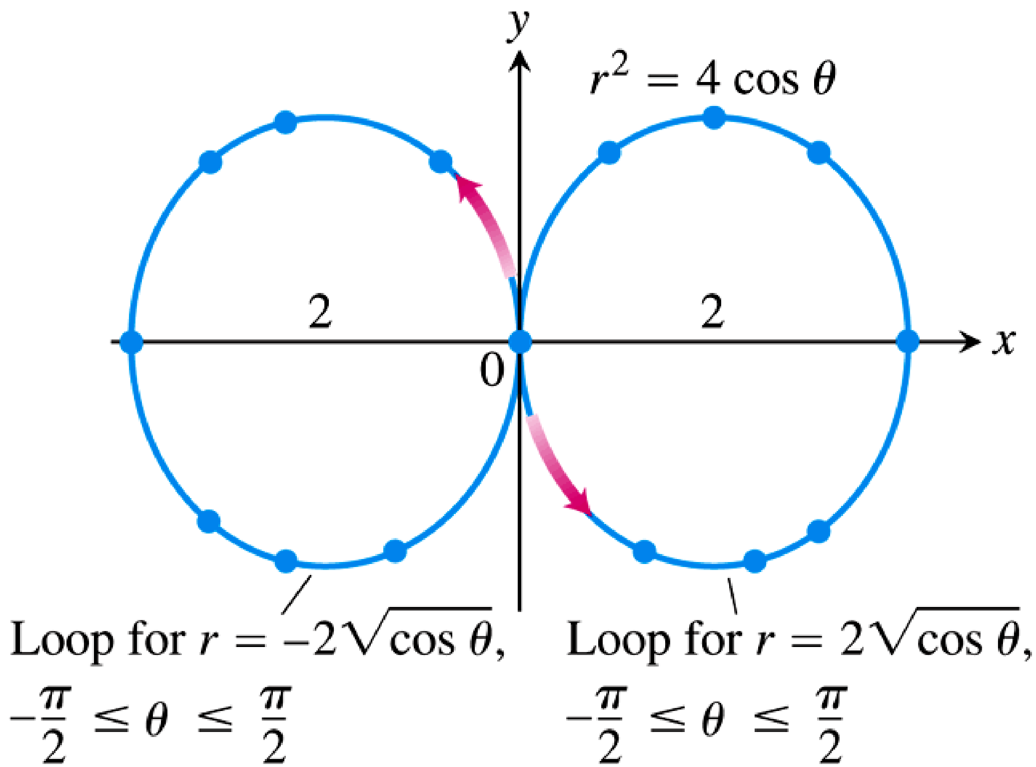


(c)

FIGURE 11.27 The steps in graphing the cardioid $r = 1 - \cos\theta$ (Example 1). The arrow shows the direction of increasing θ.

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

FIGURE 11.28 The graph of $r^2 = 4\cos\theta$. The arrows show the direction of increasing θ. The values of r in the table are rounded (Example 2).

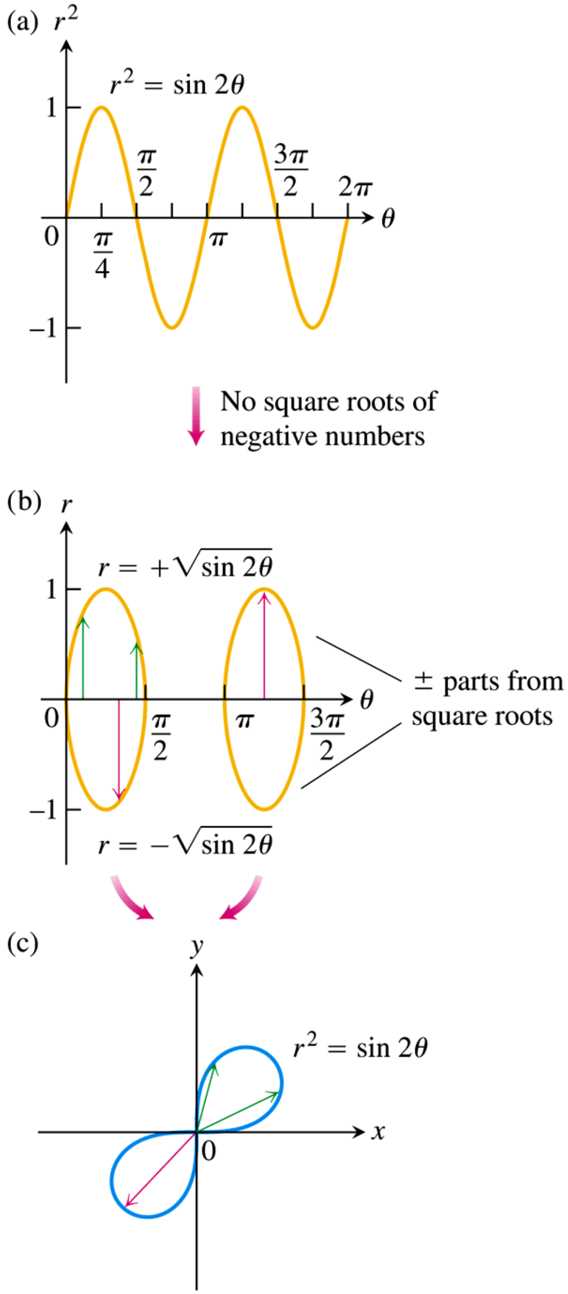


FIGURE 11.29 To plot $r = f(\theta)$ in the Cartesian $r\theta$-plane in (b), we first plot $r^2 = \sin 2\theta$ in the $r^2\theta$-plane in (a) and then ignore the values of θ for which $\sin 2\theta$ is negative. The radii from the sketch in (b) cover the polar graph of the lemniscate in (c) twice (Example 3).

11.7

Areas and Lengths in Polar Coordinates

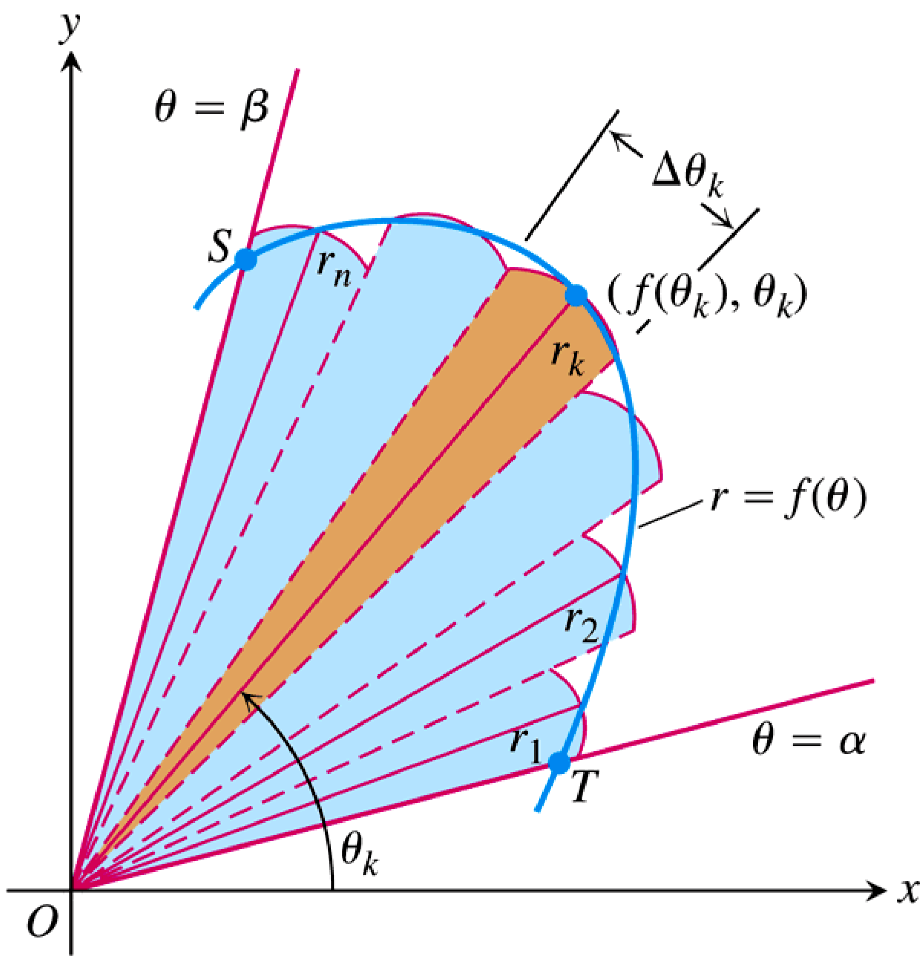


FIGURE 11.30 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

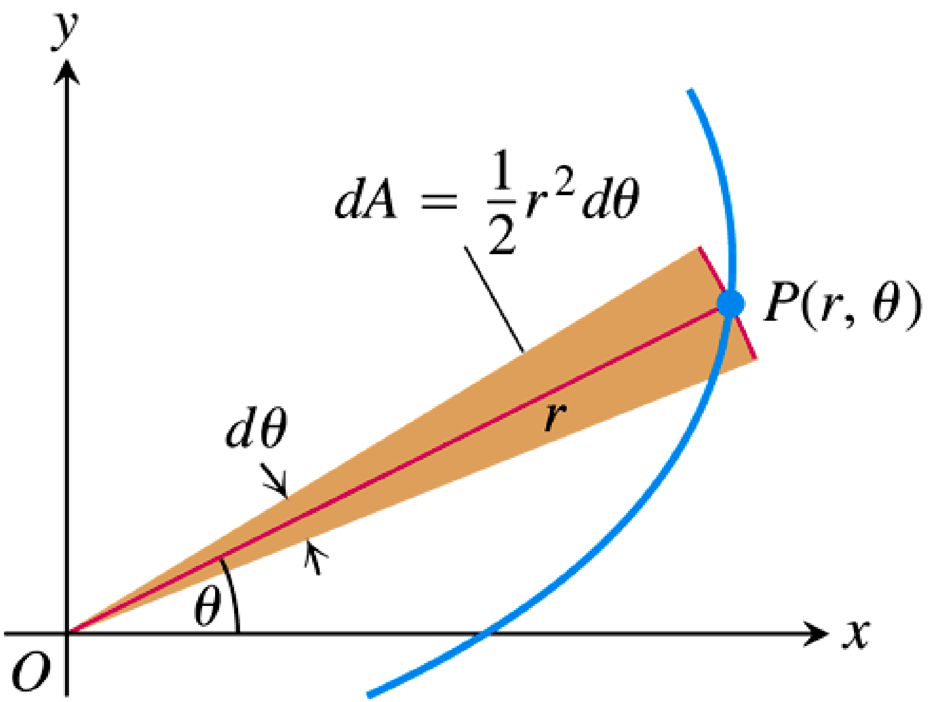


FIGURE 11.31 The area differential dA for the curve $r = f(\theta)$.

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta), \alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta .$$

This is the integral of the **area differential** (Figure 11.31)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \left(f(\theta)\right)^2 d\theta .$$

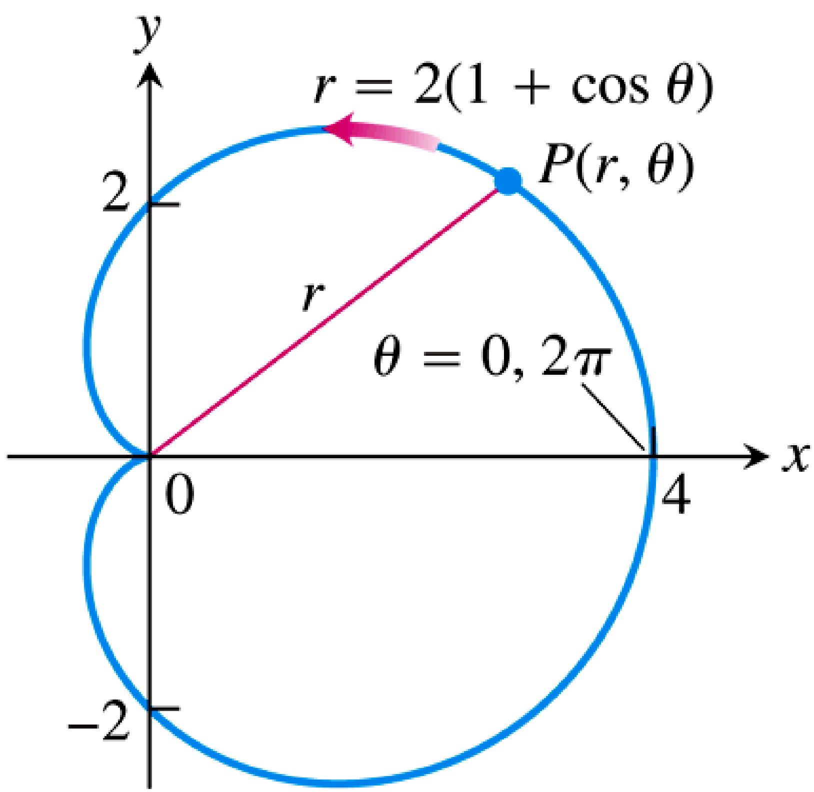


FIGURE 11.32 The cardioid in Example 1.

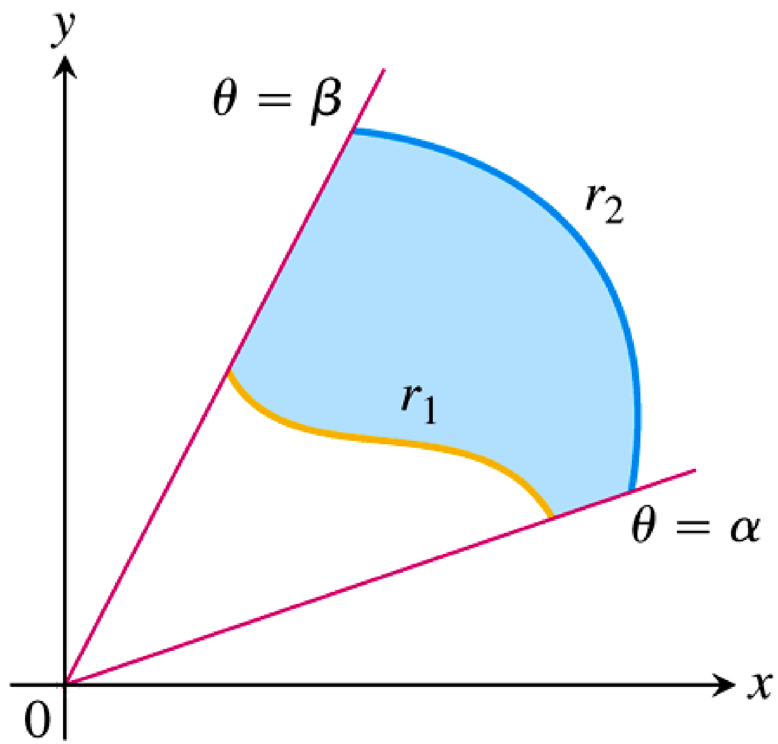


FIGURE 11.33 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

Area of the Region $0 \leq r_1(\theta) \leq r \leq r_2(\theta), \qquad \alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^{\,2}\, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^{\,2}\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right) d\theta \qquad (1)$$

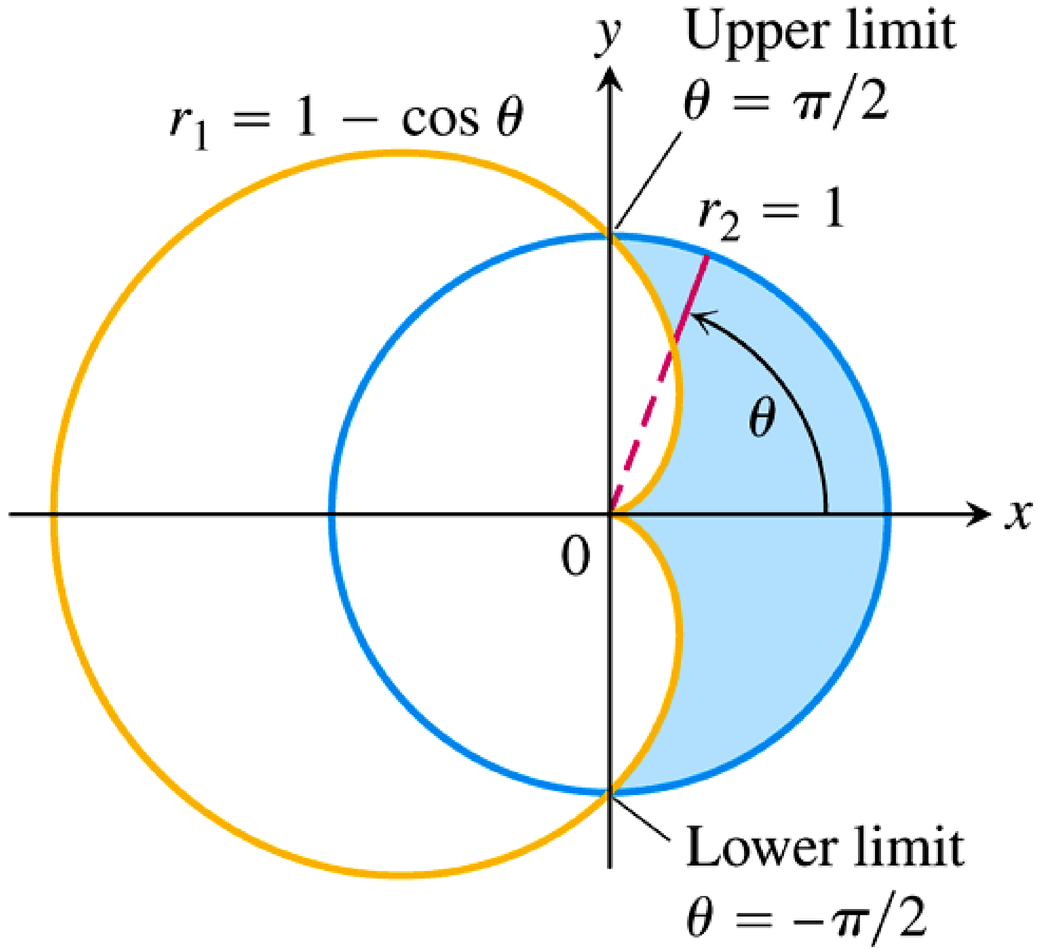


FIGURE 11.34 The region and limits of integration in Example 2.

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \qquad (3)$$

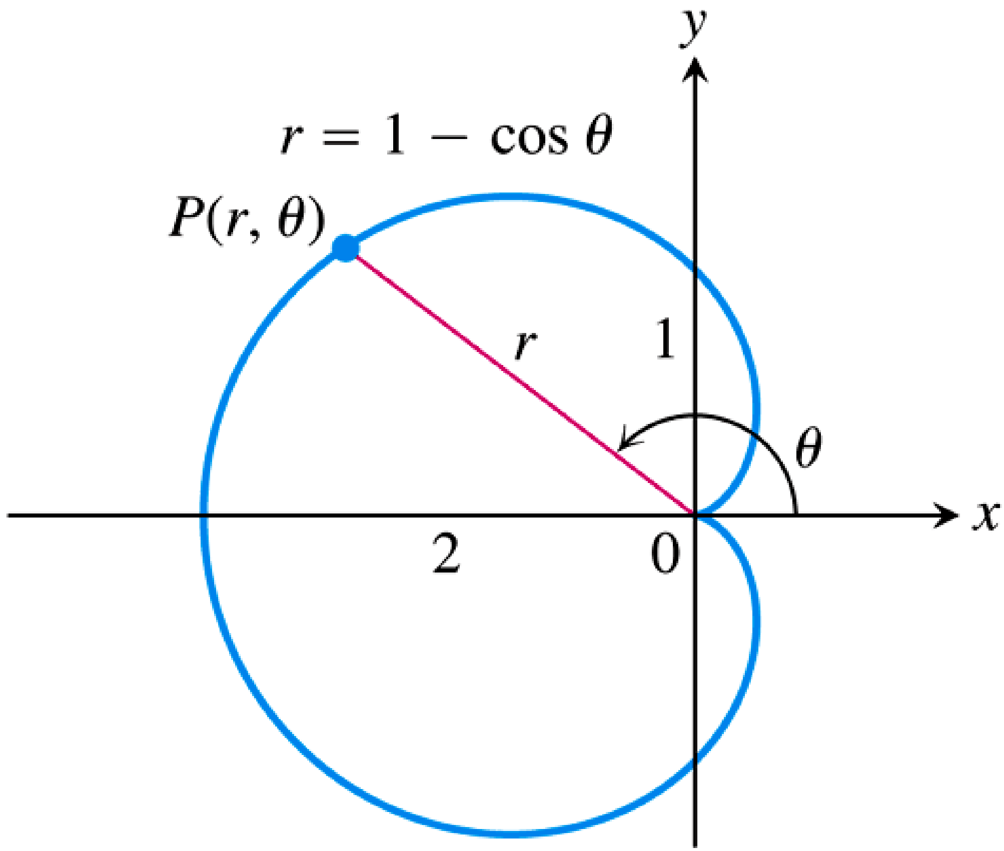
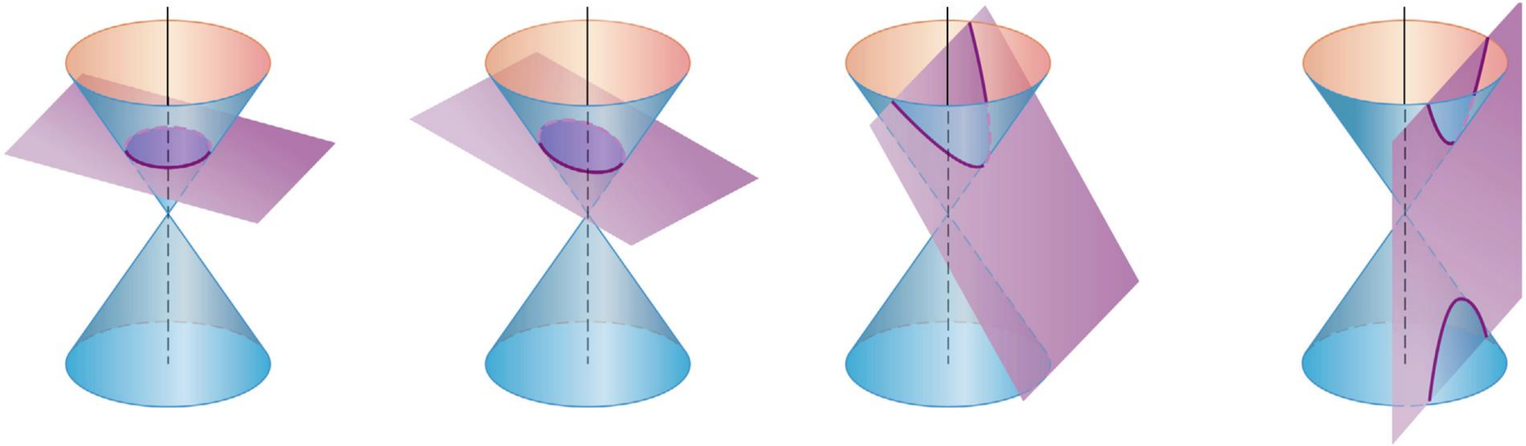


FIGURE 11.35 Calculating the length of a cardioid (Example 3).

11.6

Conic Sections

DEFINITIONS A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola**. The fixed point is the **focus** of the parabola. The fixed line is the **directrix**.



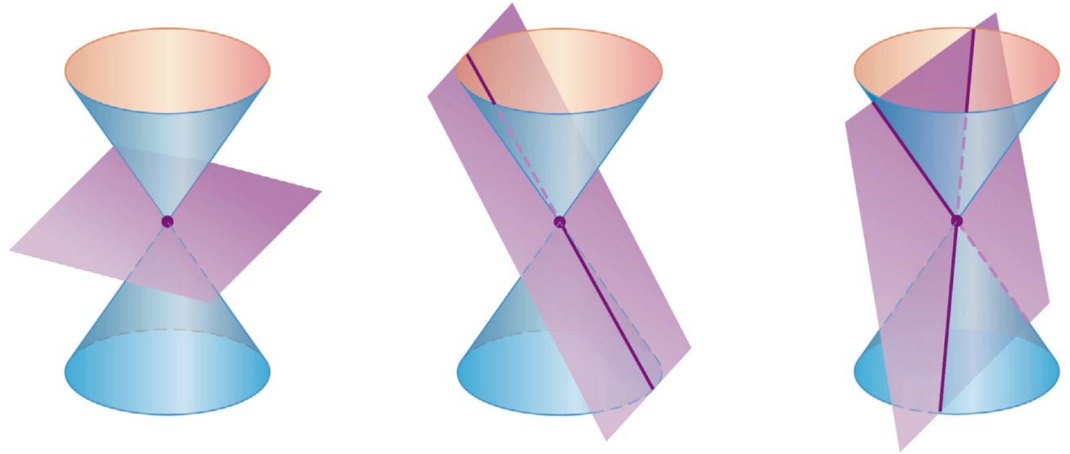
Circle: plane perpendicular to cone axis

Ellipse: plane oblique to cone axis

Parabola: plane parallel to side of cone

Hyperbola: plane cuts both halves of cone

(a)



Point: plane through cone vertex only

Single line: plane tangent to cone

Pair of intersecting lines

(b)

FIGURE 11.36 The standard conic sections (a) are the curves in which a plane cuts a *double* cone. Hyperbolas come in two parts, called *branches*. The point and lines obtained by passing the plane through the cone's vertex (b) are *degenerate* conic sections.

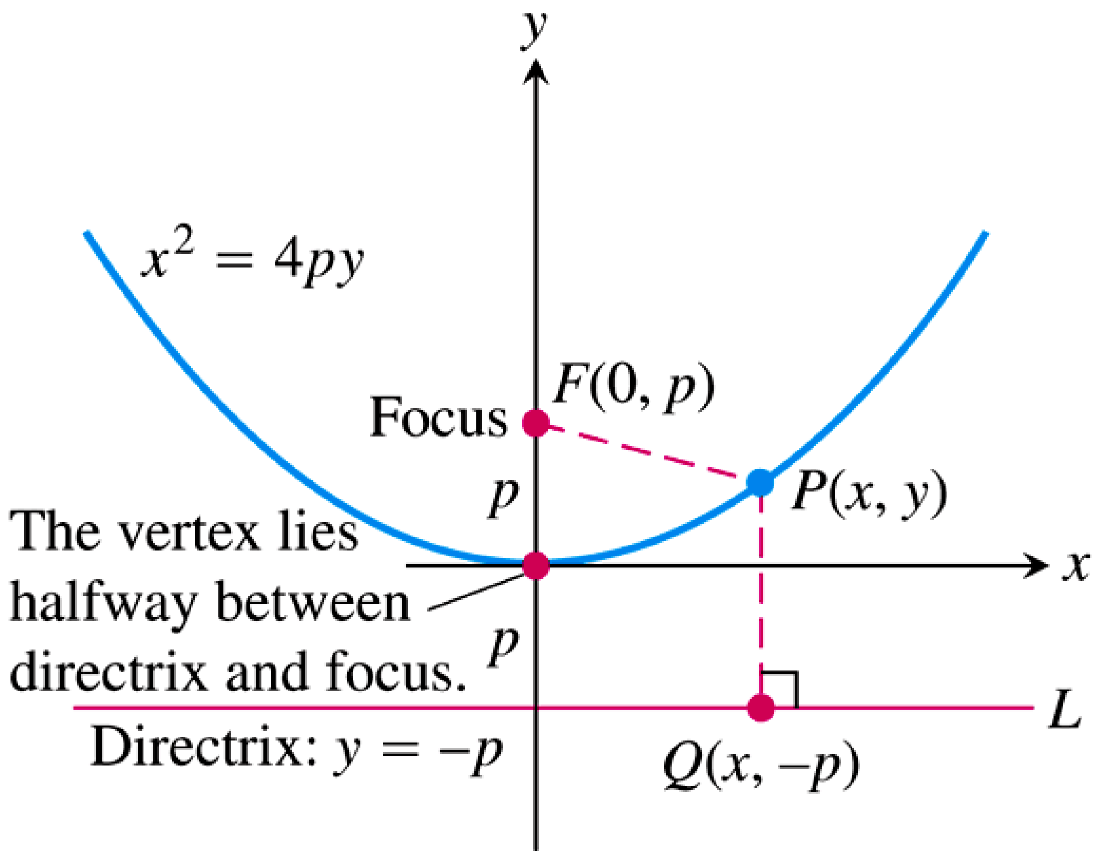


FIGURE 11.37 The standard form of the parabola $x^2 = 4py, p > 0$.

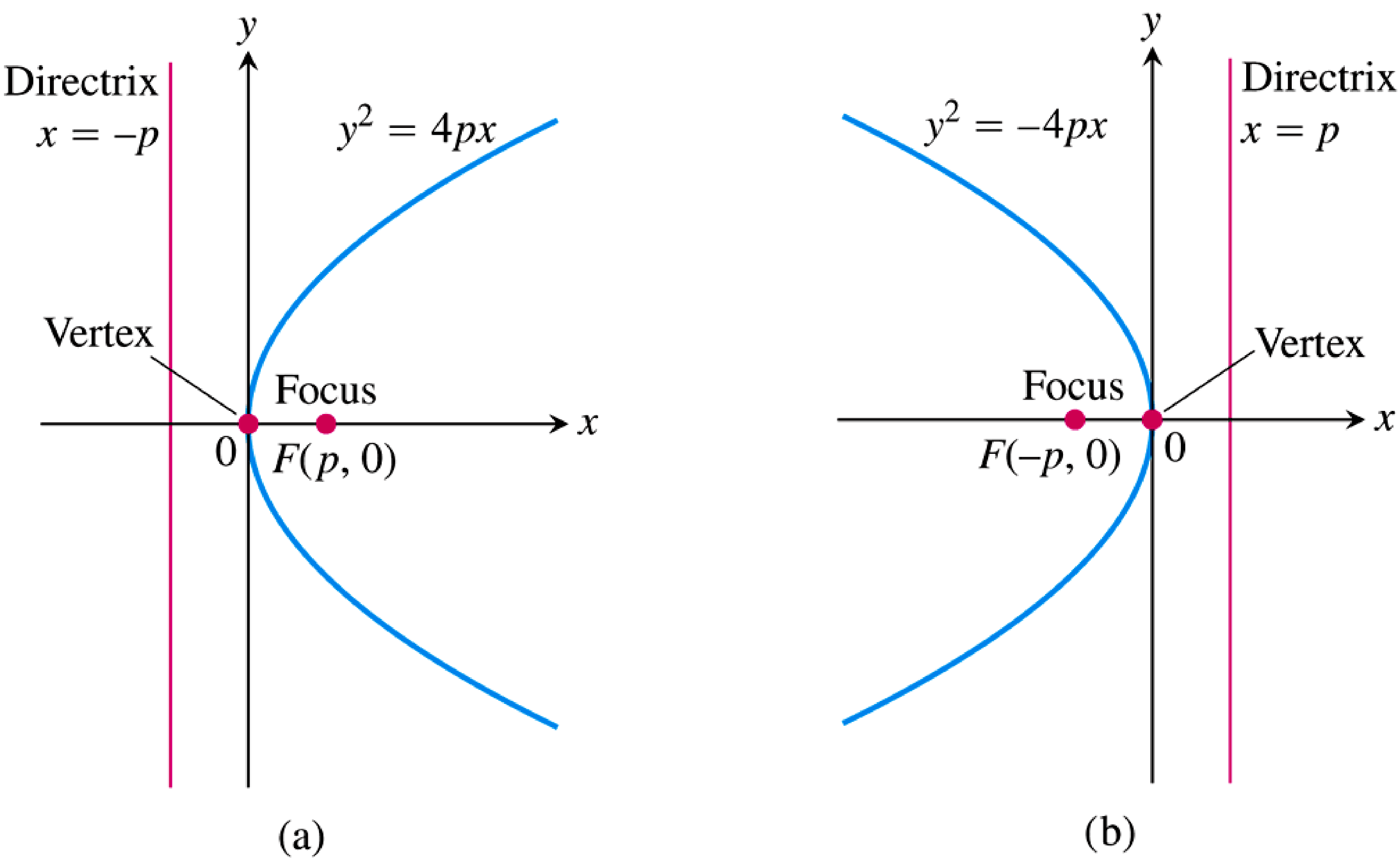


FIGURE 11.38 (a) The parabola $y^2 = 4px$. (b) The parabola $y^2 = -4px$.

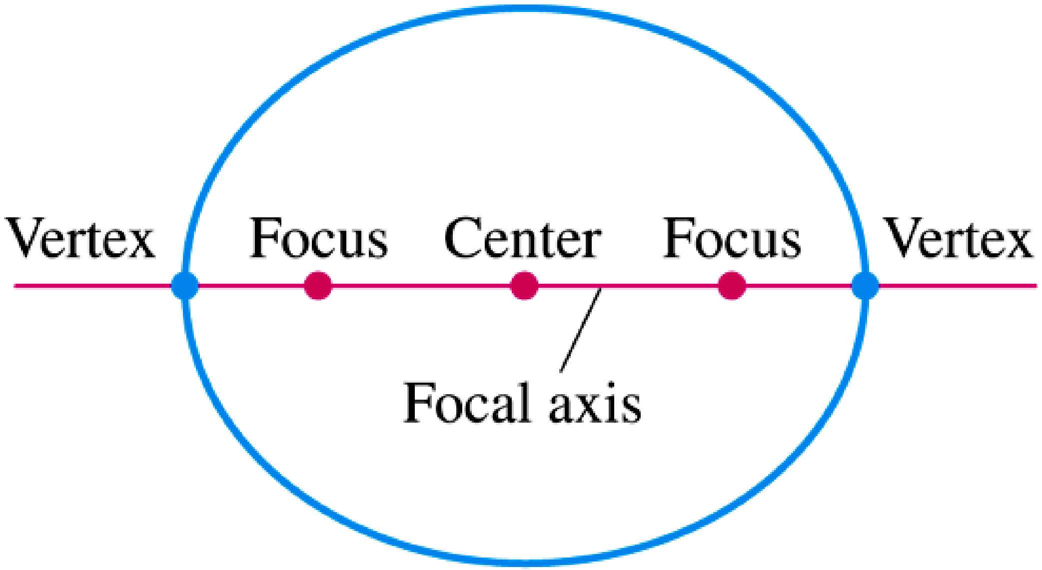


FIGURE 11.39 Points on the focal axis of an ellipse.

DEFINITIONS An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and ellipse cross are the ellipse's **vertices** (Figure 11.39).

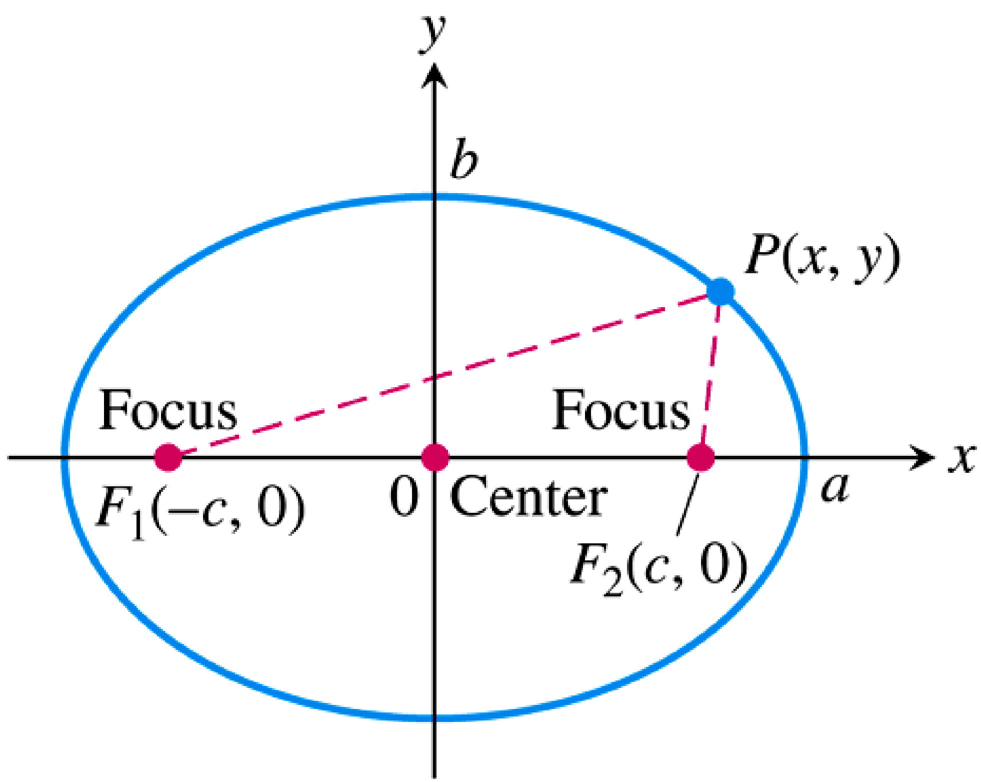


FIGURE 11.40 The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of the equation $(x^2/a^2) + (y^2/b^2) = 1$, where $b^2 = a^2 - c^2$.

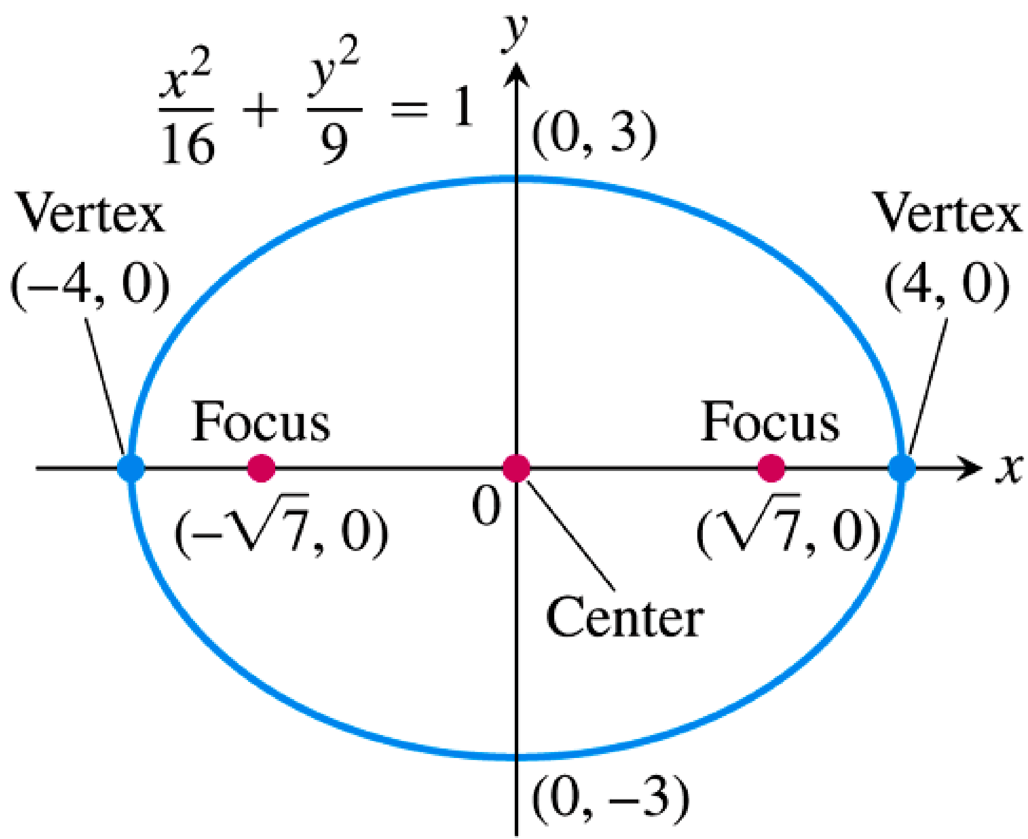


FIGURE 11.41 An ellipse with its major axis horizontal (Example 2).

Standard-Form Equations for Ellipses Centered at the Origin

Foci on the x-axis: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Foci on the y-axis: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

In each case, a is the semimajor axis and b is the semiminor axis.

DEFINITIONS A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.

The line through the foci of a hyperbola is the **focal axis**. The point on the axis halfway between the foci is the hyperbola's **center**. The points where the focal axis and hyperbola cross are the **vertices** (Figure 11.42).

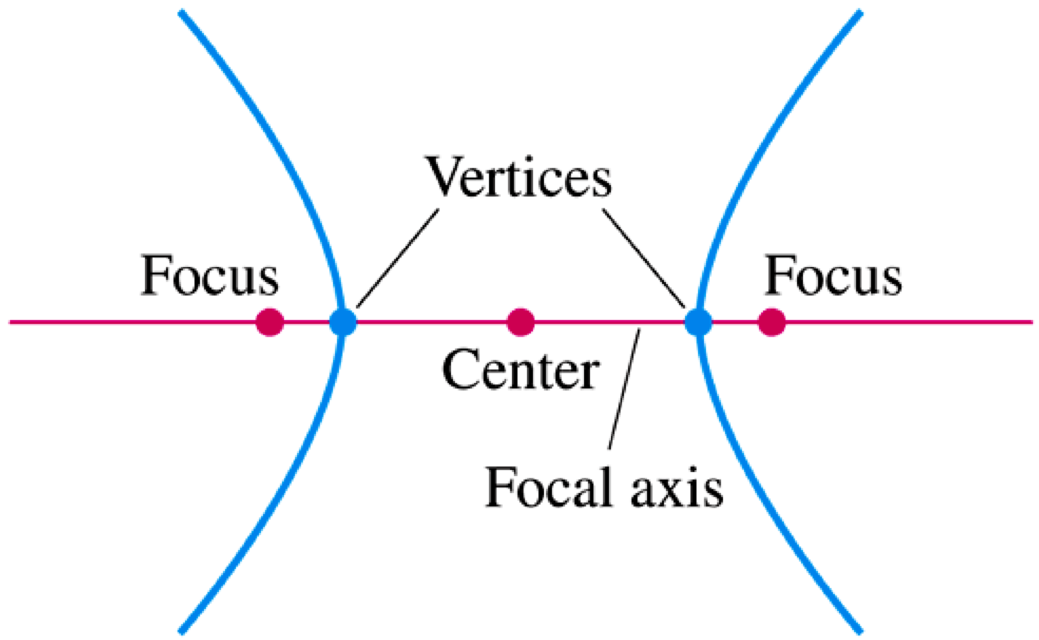


FIGURE 11.42 Points on the focal axis of a hyperbola.

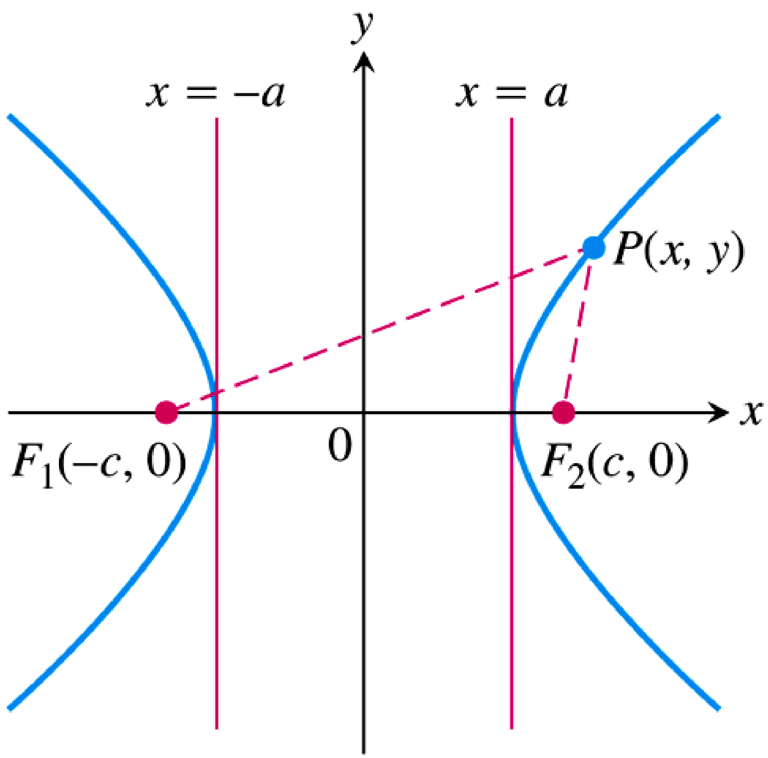


FIGURE 11.43 Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here, $PF_1 - PF_2 = 2a$. For points on the left-hand branch, $PF_2 - PF_1 = 2a$. We then let $b = \sqrt{c^2 - a^2}$.

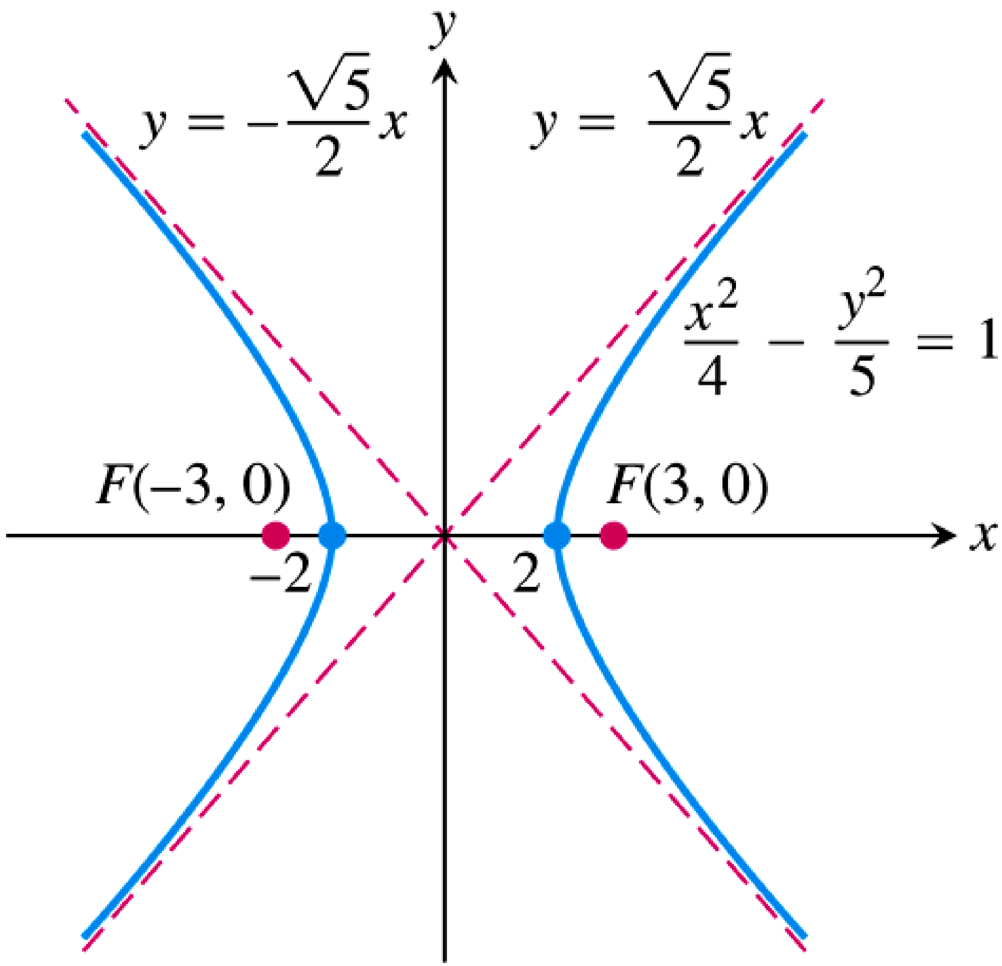


FIGURE 11.44 The hyperbola and its asymptotes in Example 3.

Standard-Form Equations for Hyperbolas Centered at the Origin

Foci on the x-axis: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Asymptotes: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0 \quad \text{or} \quad y = \pm\dfrac{b}{a}x$

Foci on the y-axis: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

Asymptotes: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 0 \quad \text{or} \quad y = \pm\dfrac{a}{b}x$

Notice the difference in the asymptote equations (b/a in the first, a/b in the second).

11.7

Conics in Polar Coordinates

DEFINITION

The **eccentricity** of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ $(a > b)$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

The **eccentricity** of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

The **eccentricity** of a parabola is $e = 1$.

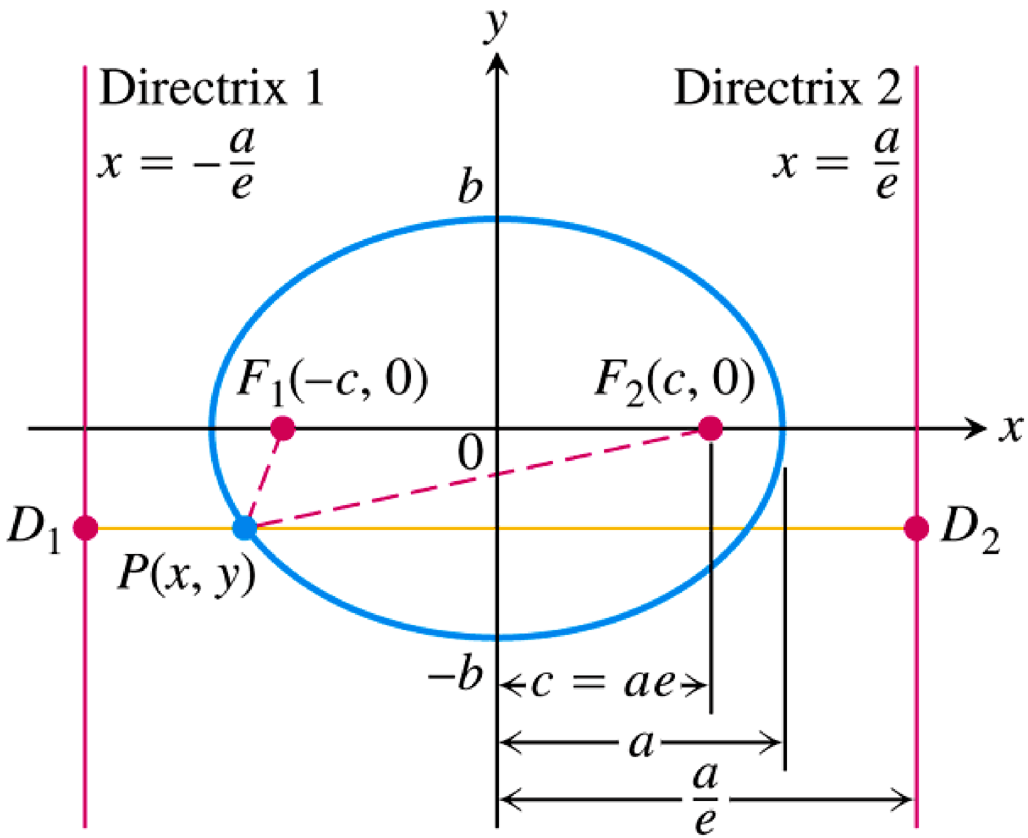


FIGURE 11.45 The foci and directrices of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Directrix 1 corresponds to focus F_1 and directrix 2 to focus F_2.

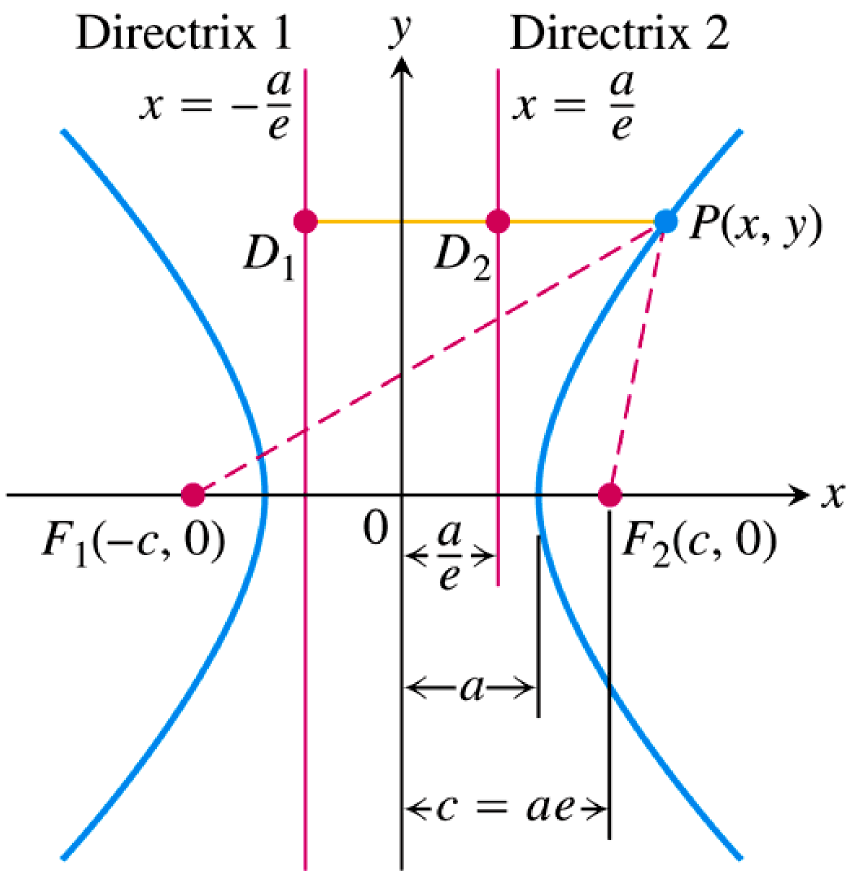


FIGURE 11.46 The foci and directrices of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$. No matter where P lies on the hyperbola, $PF_1 = e \cdot PD_1$ and $PF_2 = e \cdot PD_2$.

In both the ellipse and the hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices (because $c/a = 2c/2a$).

$$\boxed{\text{Eccentricity} = \frac{\text{distance between foci}}{\text{distance between vertices}}}$$

The “focus–directrix” equation $PF = e \cdot PD$ unites the parabola, ellipse, and hyperbola in the following way. Suppose that the distance PF of a point P from a fixed point F (the focus) is a constant multiple of its distance from a fixed line (the directrix). That is, suppose

$$\boxed{PF = e \cdot PD, \qquad (4)}$$

where e is the constant of proportionality. Then the path traced by P is

(a) a *parabola* if $e = 1$,

(b) an *ellipse* of eccentricity e if $e < 1$, and

(c) a *hyperbola* of eccentricity e if $e > 1$.

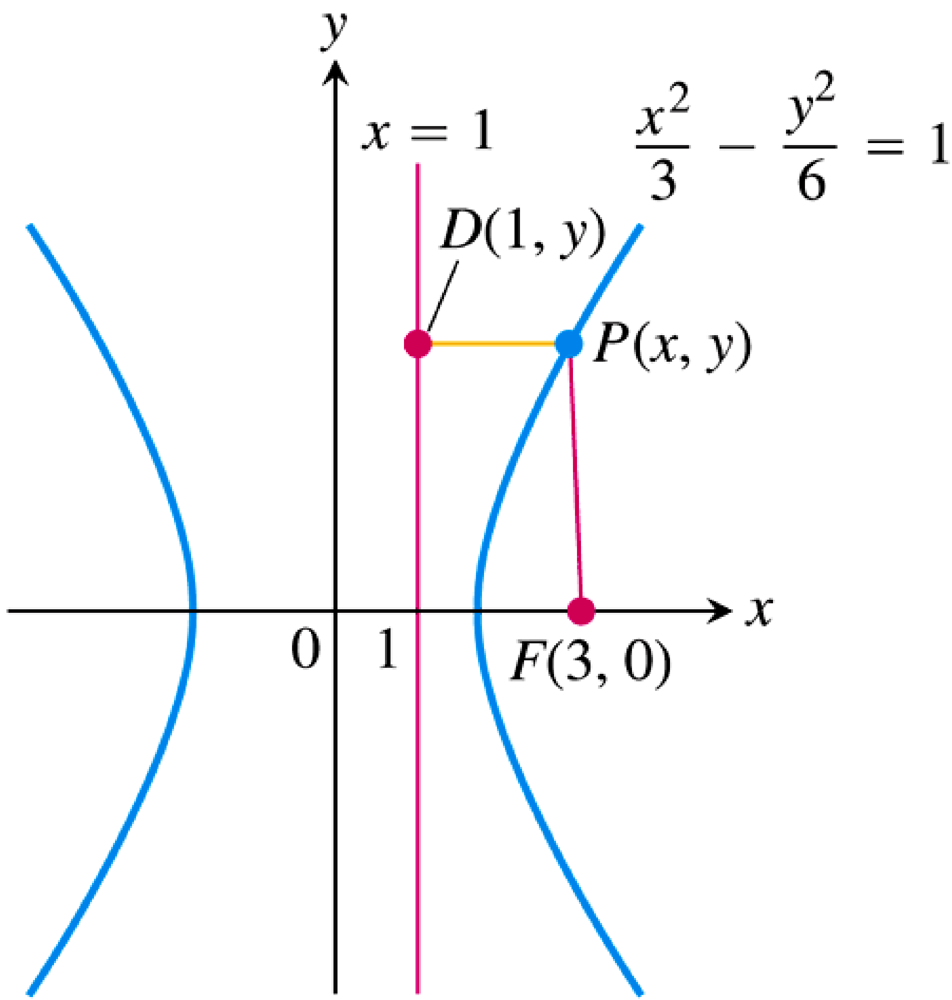


FIGURE 11.47 The hyperbola and directrix in Example 1.

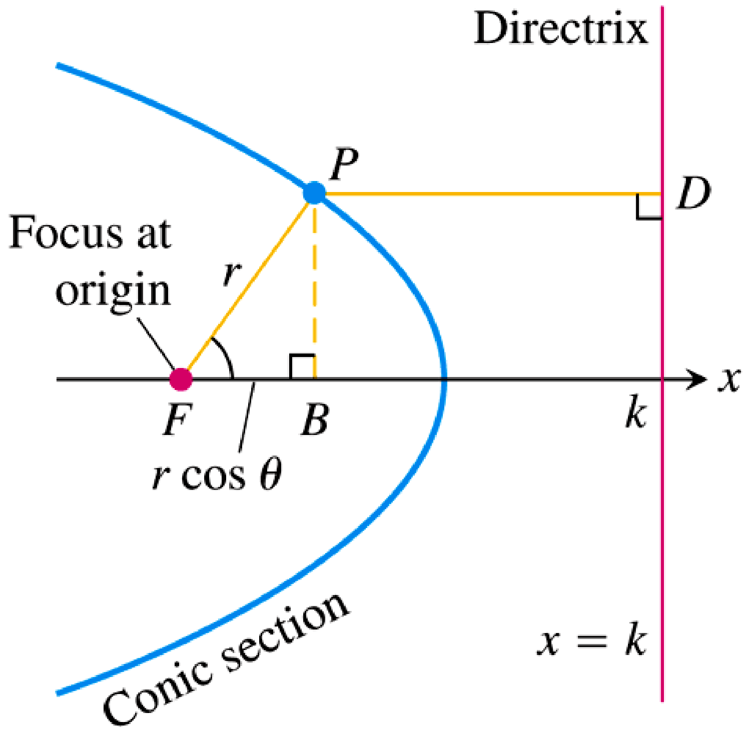


FIGURE 11.48 If a conic section is put in the position with its focus placed at the origin and a directrix perpendicular to the initial ray and right of the origin, we can find its polar equation from the conic's focus–directrix equation.

Polar Equation for a Conic with Eccentricity *e*

$$r = \frac{ke}{1 + e\cos\theta}, \qquad (5)$$

where $x = k > 0$ is the vertical directrix.

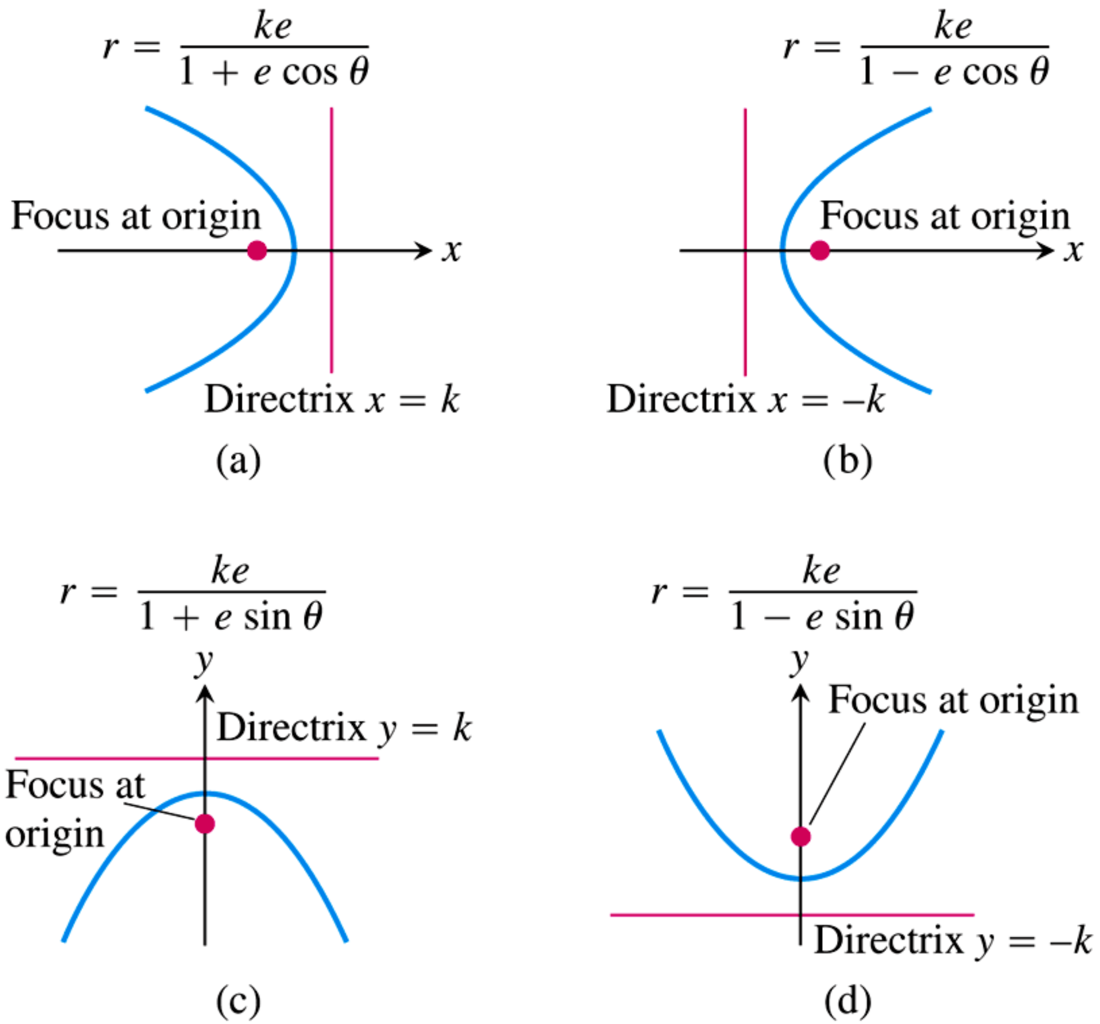


FIGURE 11.49 Equations for conic sections with eccentricity $e > 0$ but different locations of the directrix. The graphs here show a parabola, so $e = 1$.

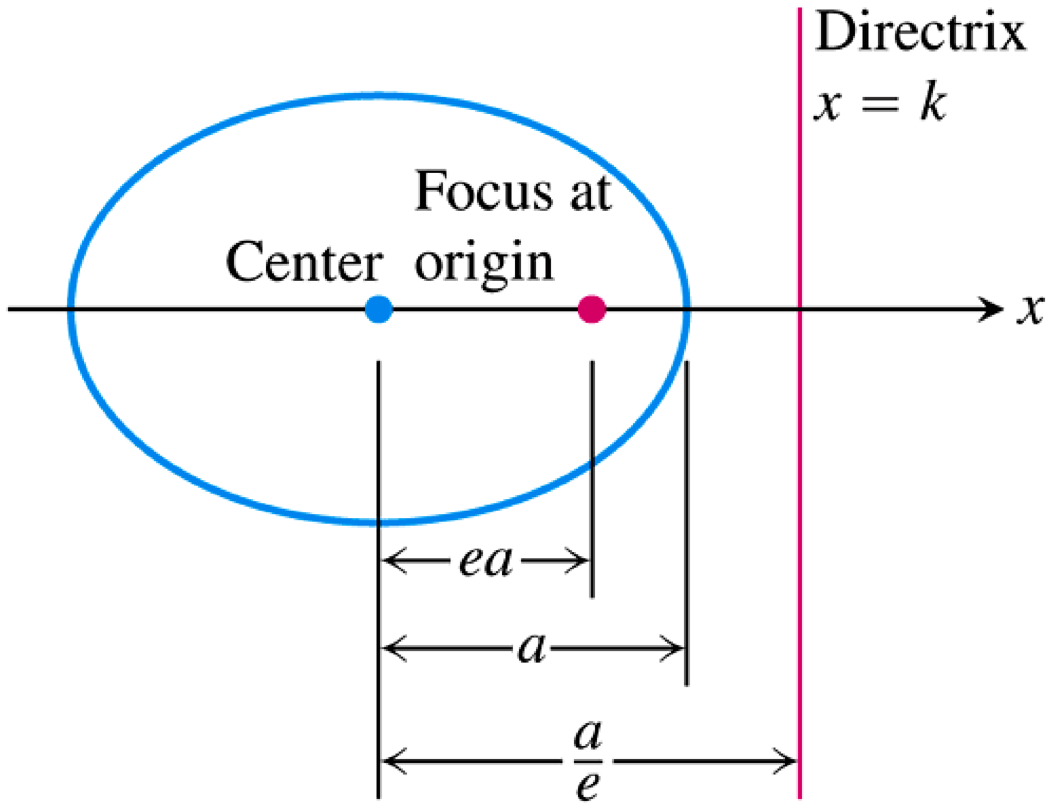


FIGURE 11.50 In an ellipse with semimajor axis a, the focus–directrix distance is $k = (a/e) - ea$, so $ke = a(1 - e^2)$.

Polar Equation for the Ellipse with Eccentricity *e* and Semimajor Axis *a*

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \tag{6}$$

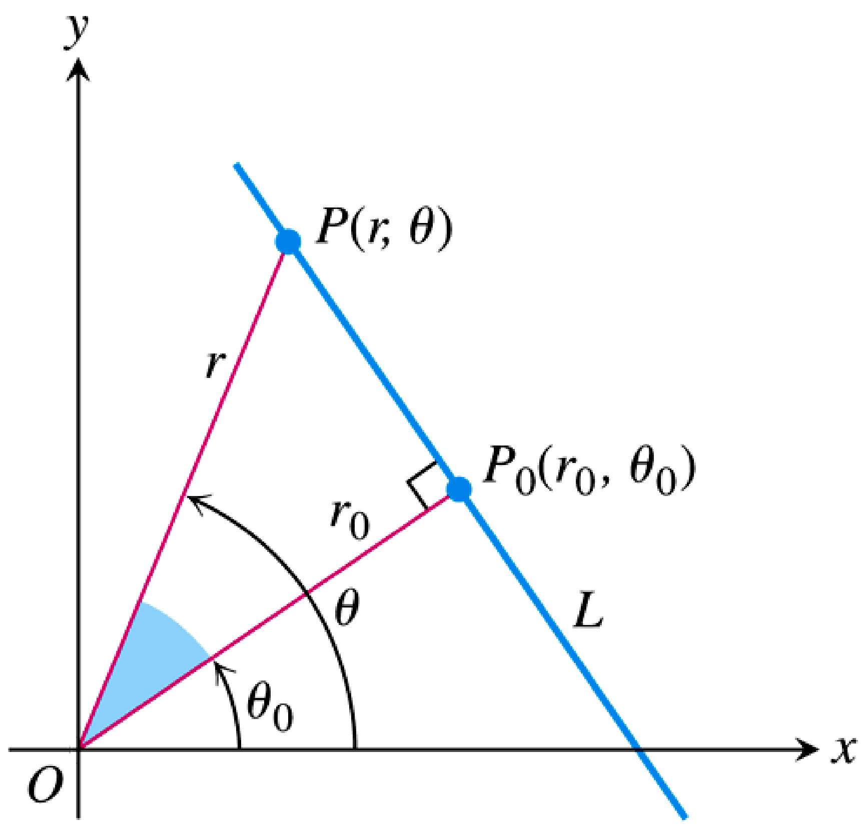


FIGURE 11.51 We can obtain a polar equation for line L by reading the relation $r_0 = r\cos(\theta - \theta_0)$ from the right triangle OP_0P.

The Standard Polar Equation for Lines

If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line L, and $r_0 \geq 0$, then an equation for L is

$$r \cos(\theta - \theta_0) = r_0. \tag{7}$$

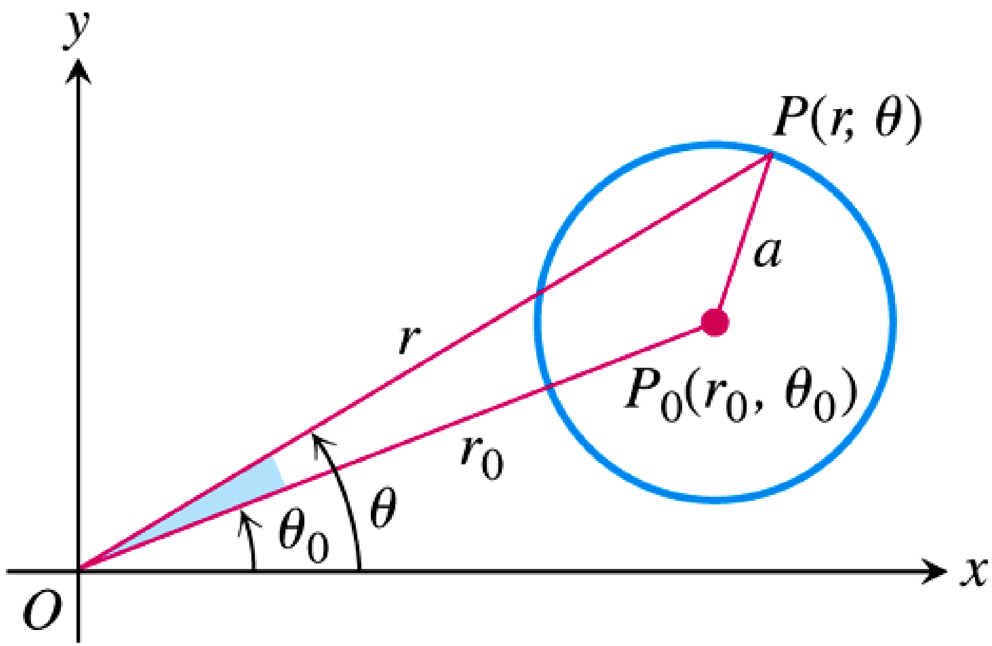


FIGURE 11.52 We can get a polar equation for this circle by applying the Law of Cosines to triangle OP_0P.

EXAMPLE 5 Here are several polar equations given by Equations (8) and (9) for circles through the origin and having centers that lie on the x- or y-axis.

Radius	Center (polar coordinates)	Polar equation
3	$(3, 0)$	$r = 6\cos\theta$
2	$(2, \pi/2)$	$r = 4\sin\theta$
1/2	$(-1/2, 0)$	$r = -\cos\theta$
1	$(-1, \pi/2)$	$r = -2\sin\theta$