

Chapter 1

Functions

1.1

Functions and Their Graphs

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

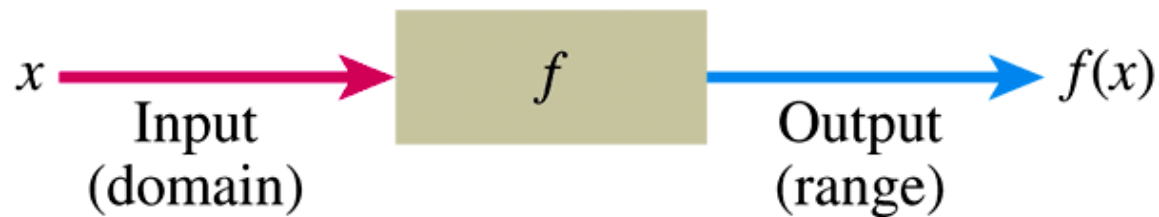
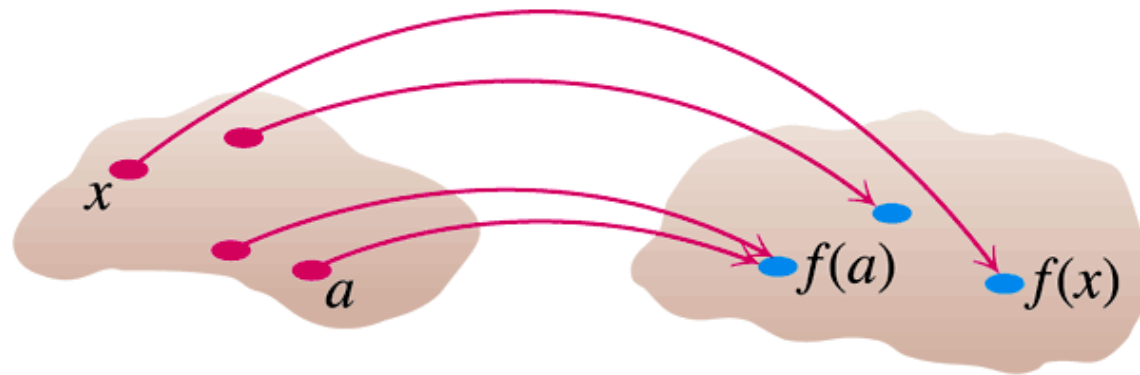


FIGURE 1.1 A diagram showing a function as a kind of machine.



D = domain set

Y = set containing
the range

FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D .

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

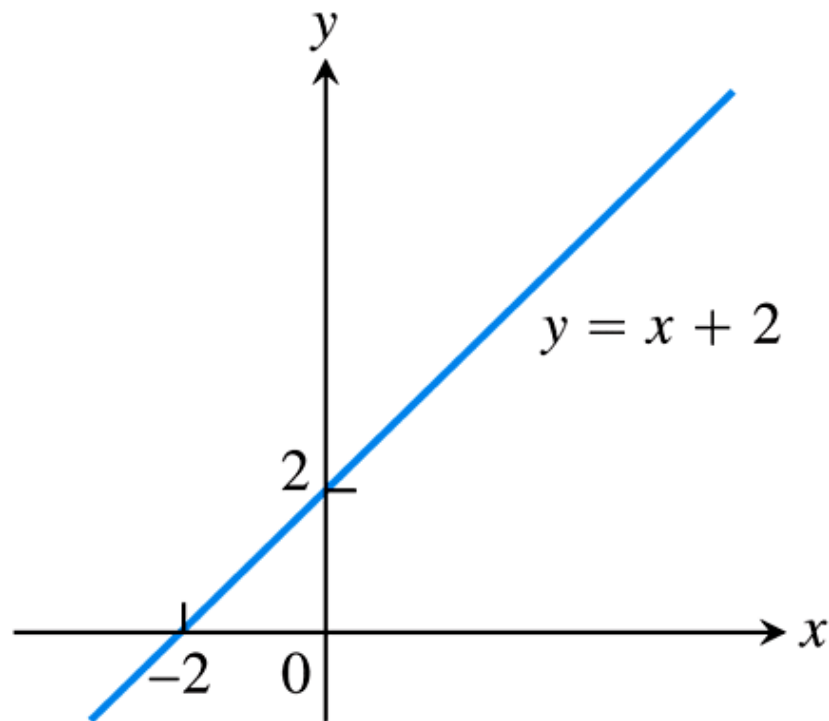


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

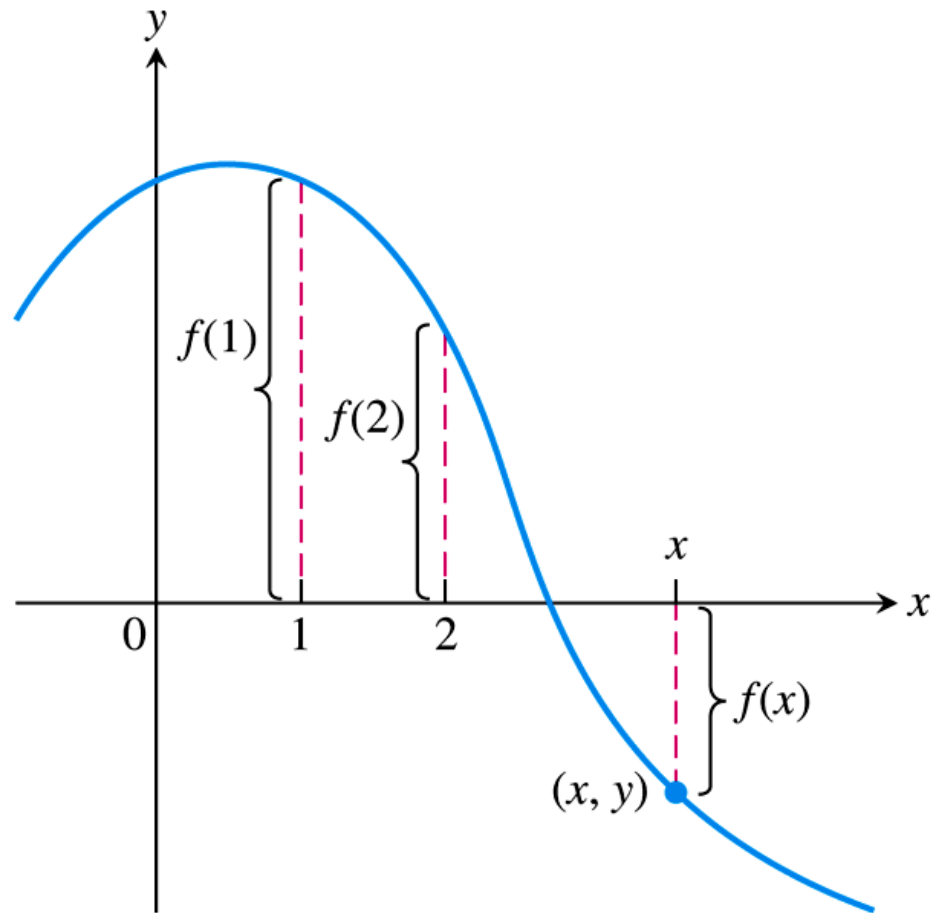


FIGURE 1.4 If (x, y) lies on the graph of f , then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

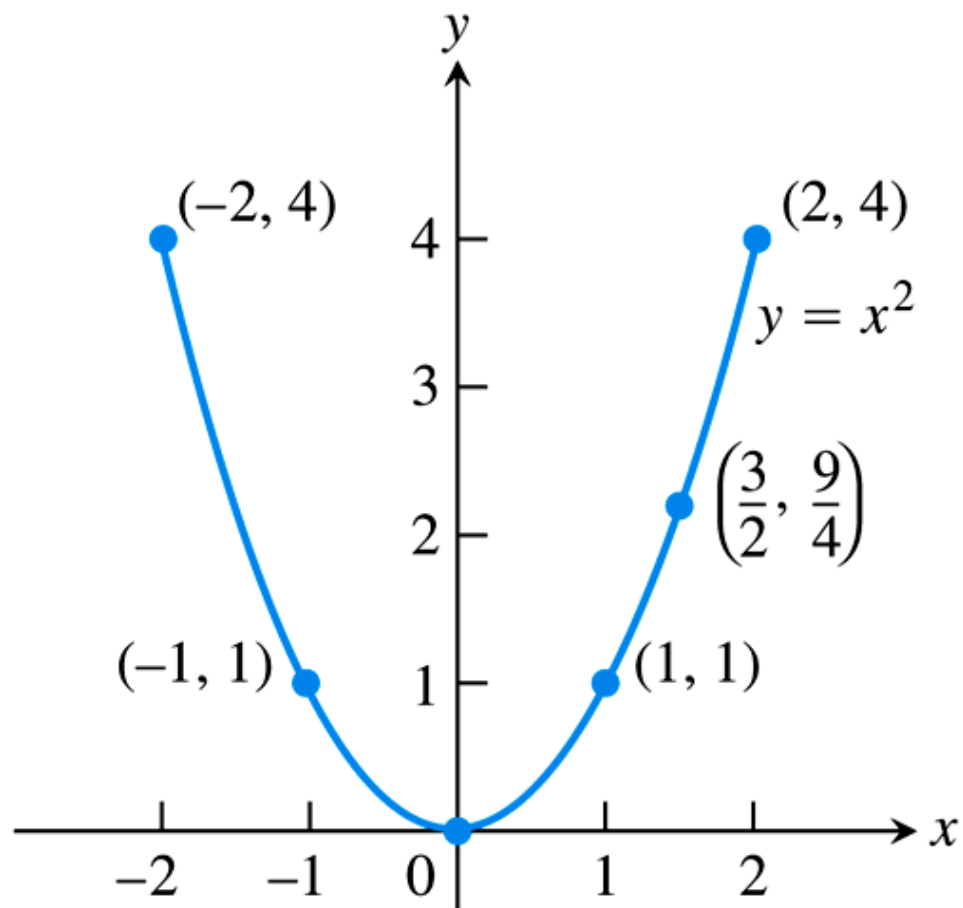


FIGURE 1.5 Graph of the function in Example 2.

TABLE 1.1 Tuning fork data

Time	Pressure	Time	Pressure
0.00091	−0.080	0.00362	0.217
0.00108	0.200	0.00379	0.480
0.00125	0.480	0.00398	0.681
0.00144	0.693	0.00416	0.810
0.00162	0.816	0.00435	0.827
0.00180	0.844	0.00453	0.749
0.00198	0.771	0.00471	0.581
0.00216	0.603	0.00489	0.346
0.00234	0.368	0.00507	0.077
0.00253	0.099	0.00525	−0.164
0.00271	−0.141	0.00543	−0.320
0.00289	−0.309	0.00562	−0.354
0.00307	−0.348	0.00579	−0.248
0.00325	−0.248	0.00598	−0.035
0.00344	−0.041		

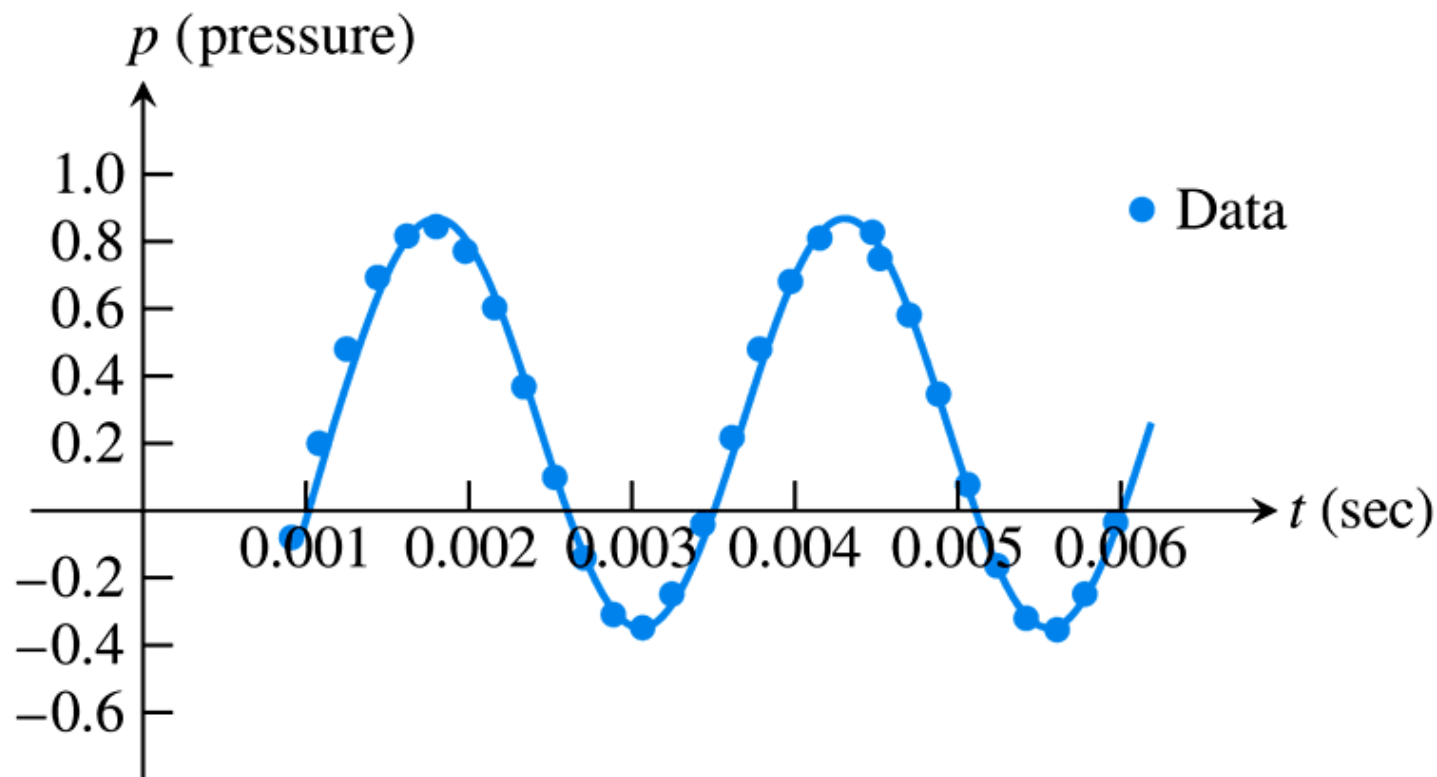
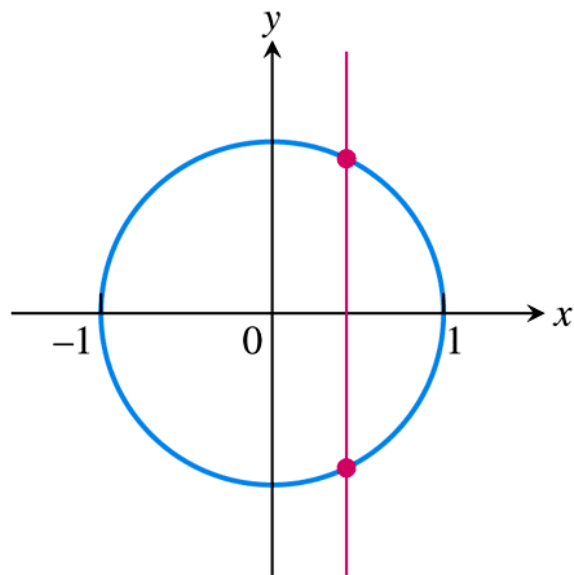
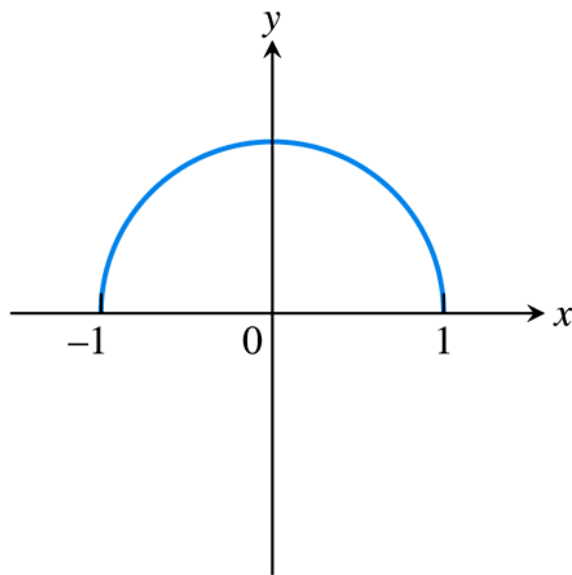


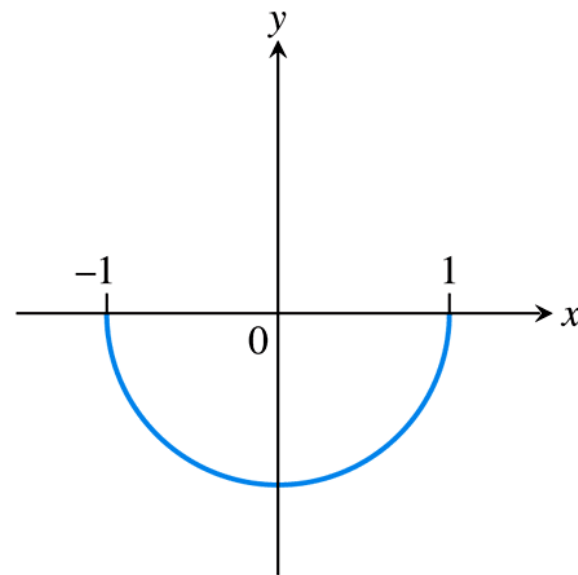
FIGURE 1.6 A smooth curve through the plotted points gives a graph of the pressure function represented by Table 1.1 (Example 3).



(a) $x^2 + y^2 = 1$



(b) $y = \sqrt{1 - x^2}$



(c) $y = -\sqrt{1 - x^2}$

FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

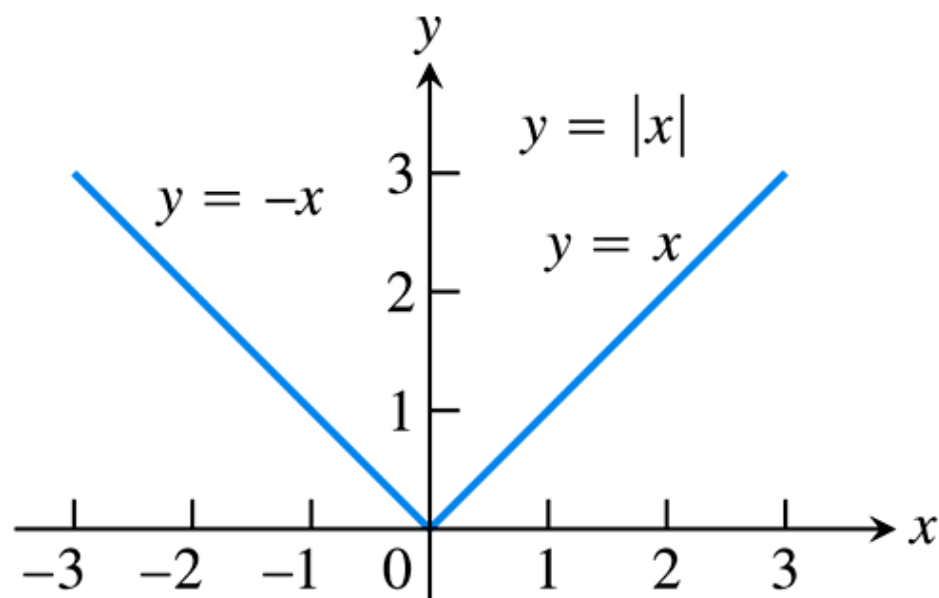


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

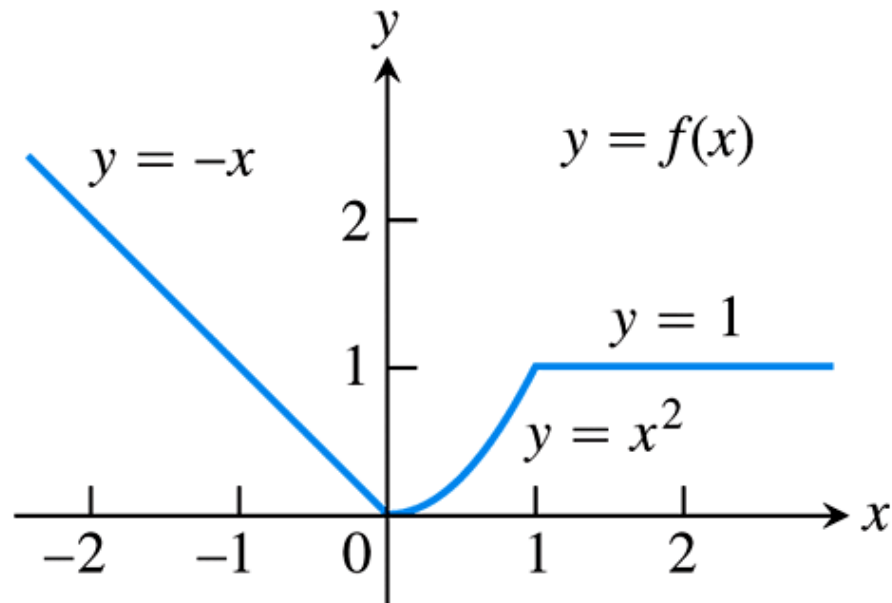


FIGURE 1.9 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 4).

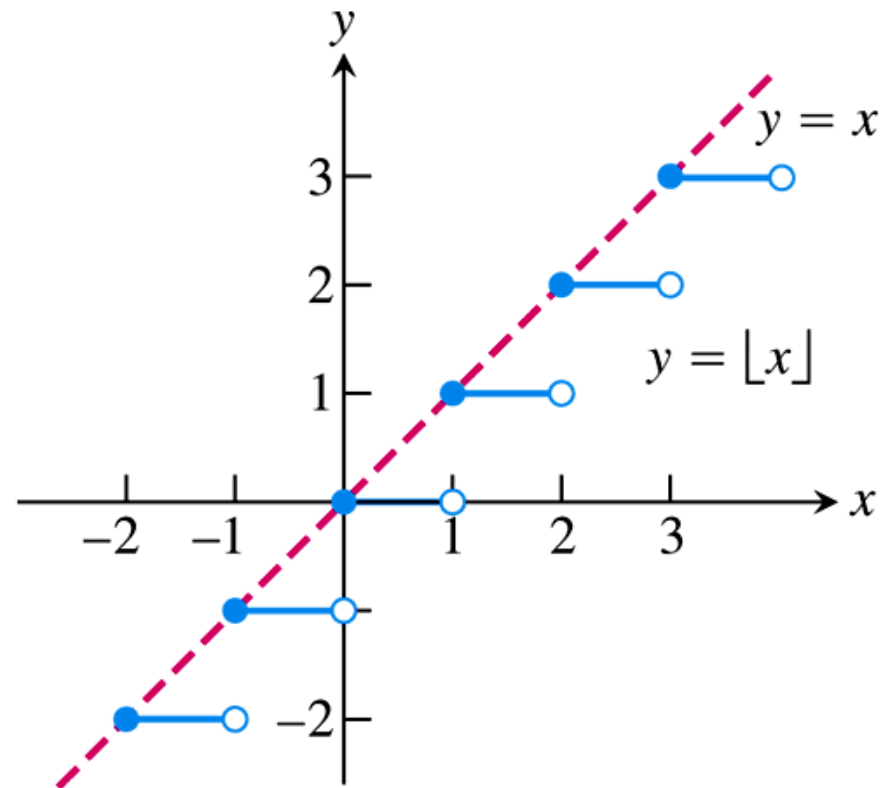


FIGURE 1.10 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 5).

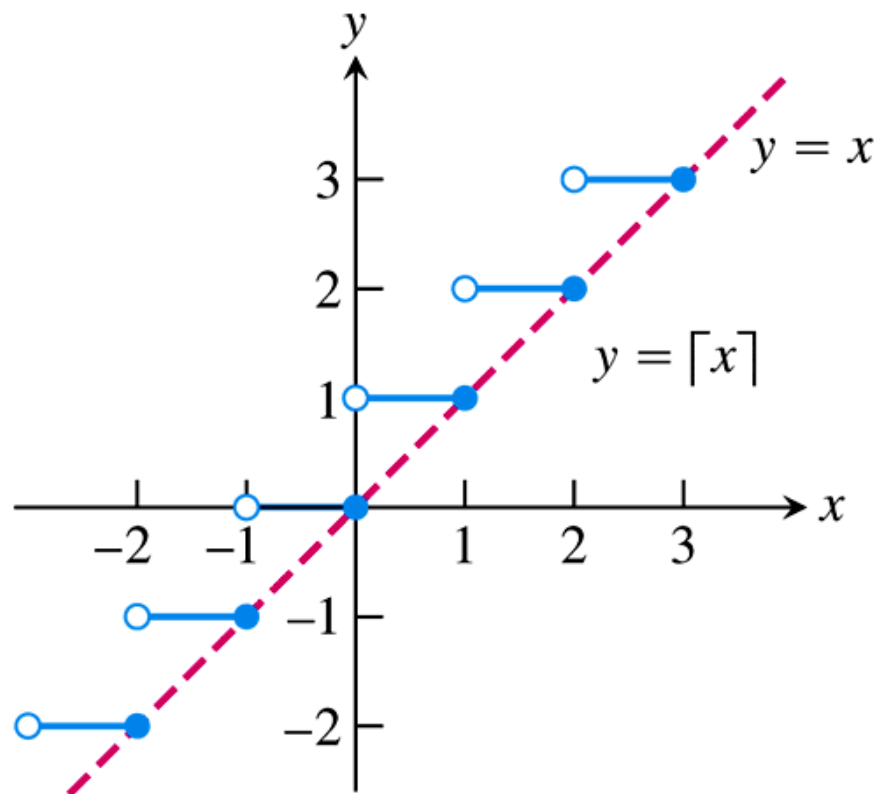


FIGURE 1.11 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 6).

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I .

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I .
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I .

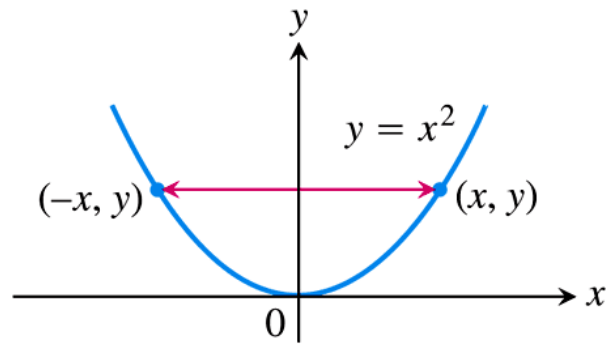
DEFINITIONS

A function $y = f(x)$ is an

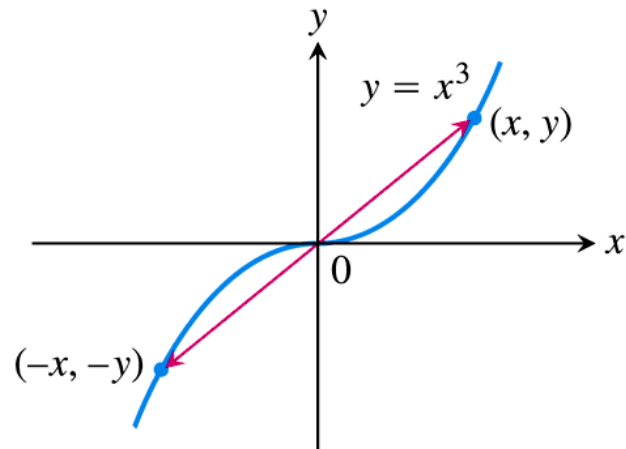
even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

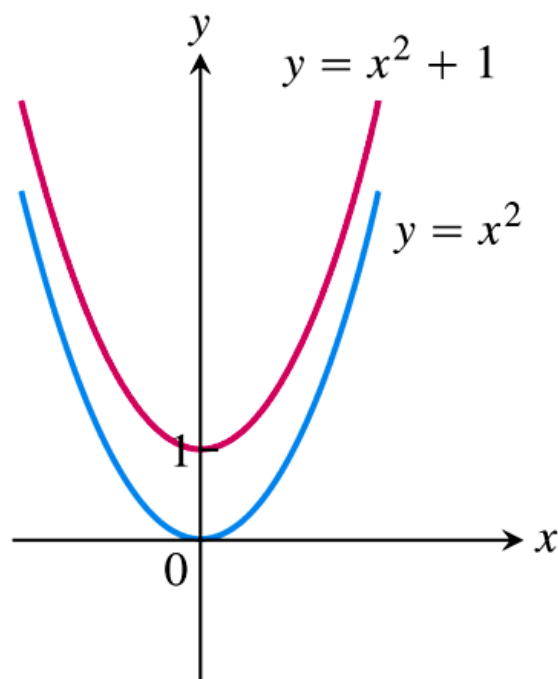


(a)

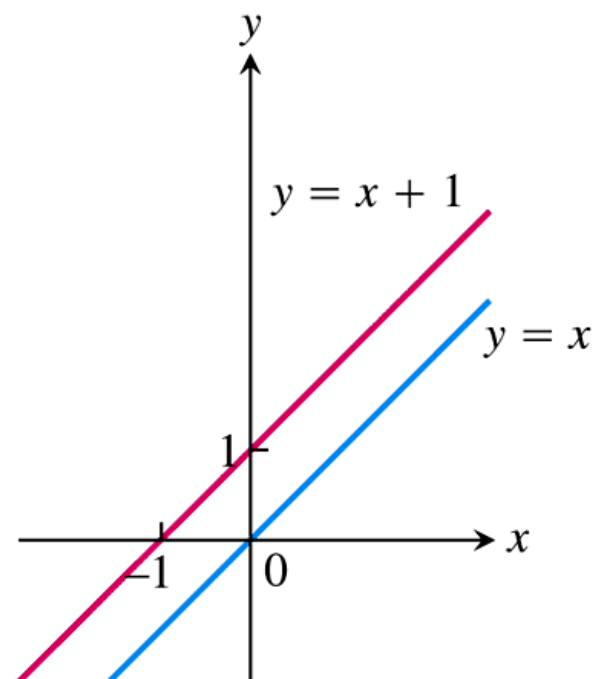


(b)

FIGURE 1.12 (a) The graph of $y = x^2$ (an even function) is symmetric about the y -axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.



(a)



(b)

FIGURE 1.13 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost (Example 8).

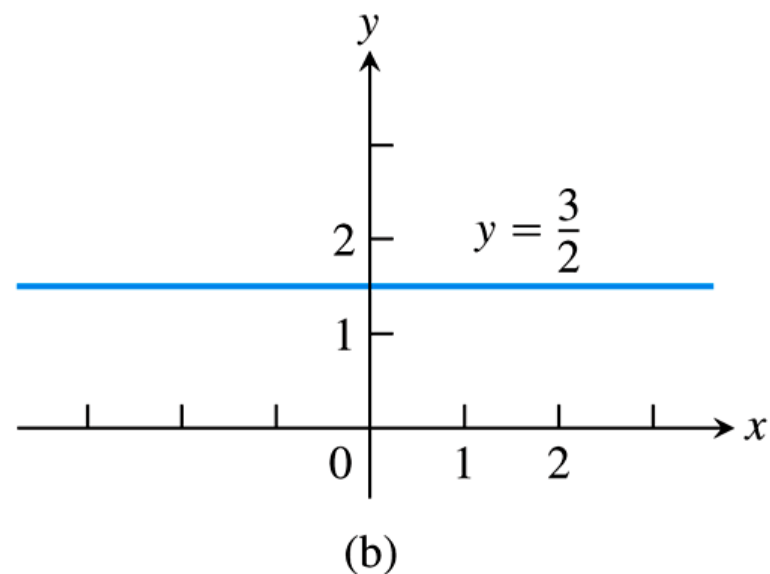
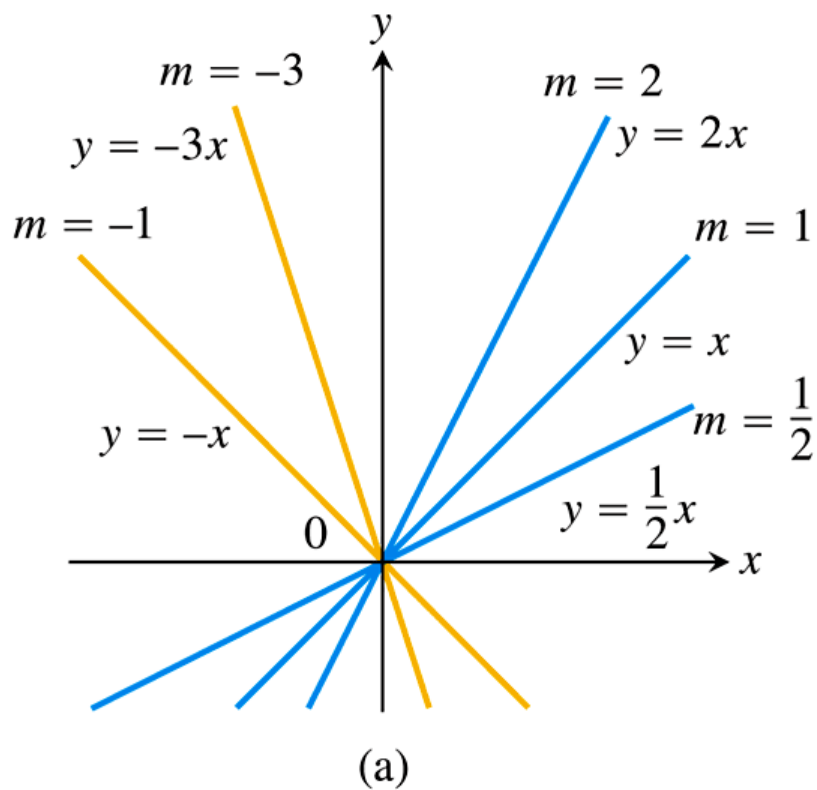


FIGURE 1.14 (a) Lines through the origin with slope m . (b) A constant function with slope $m = 0$.

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if $y = kx$ for some nonzero constant k .

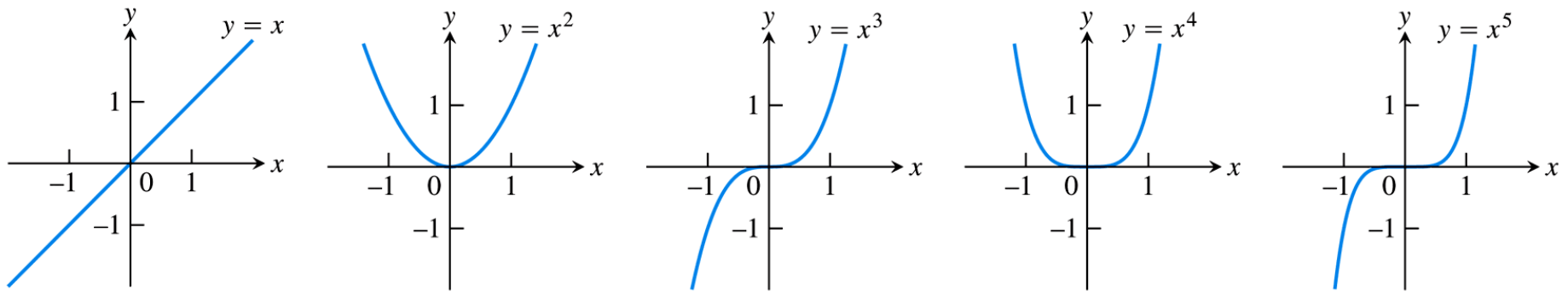


FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

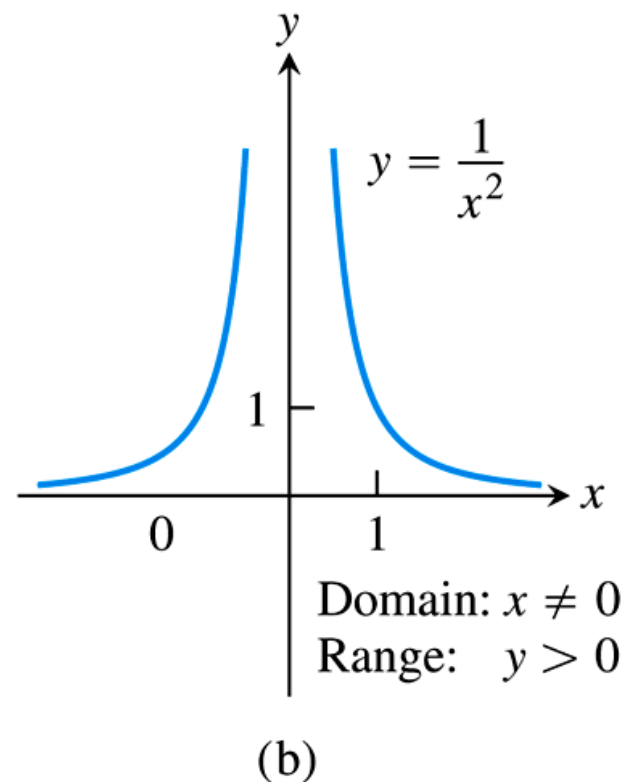
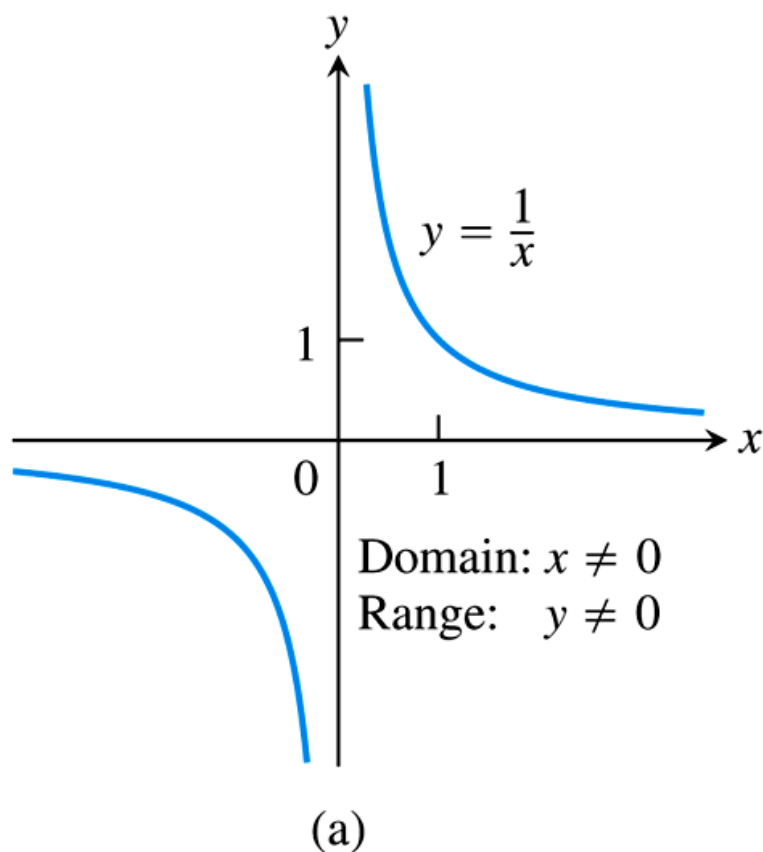


FIGURE 1.16 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

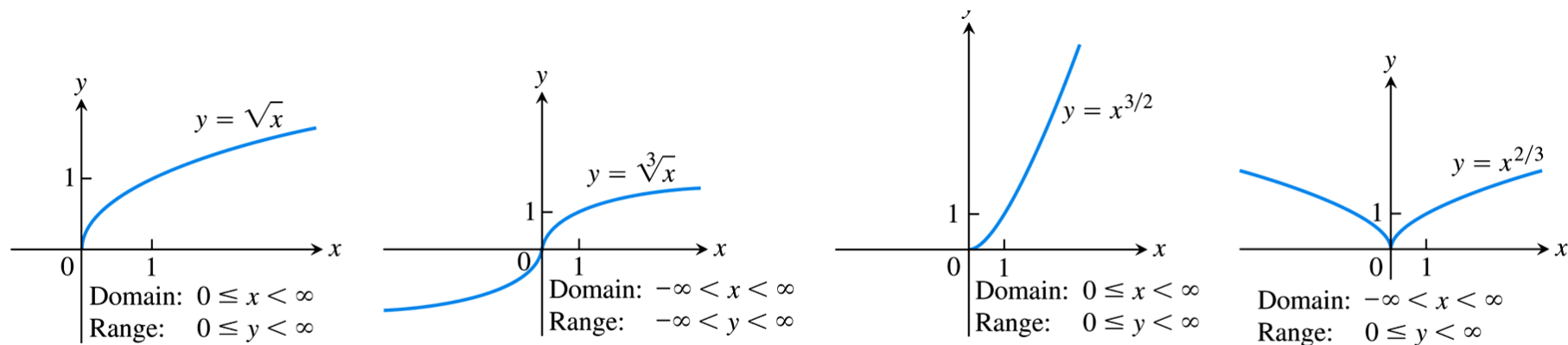


FIGURE 1.17 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2},$ and $\frac{2}{3}$.

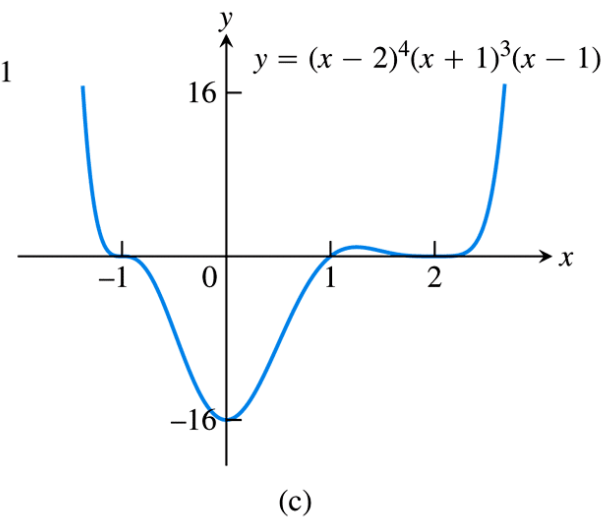
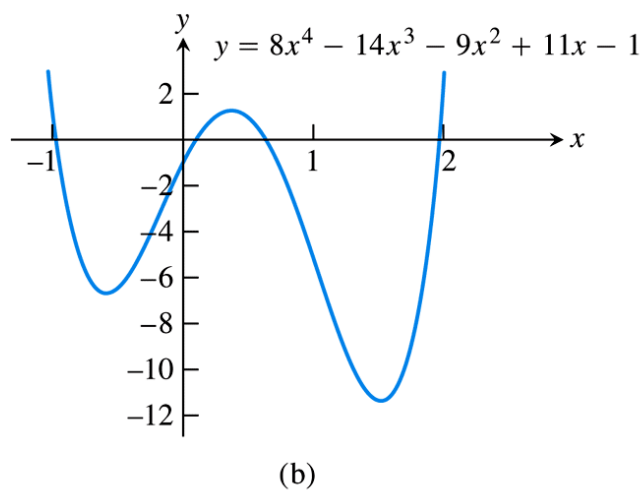
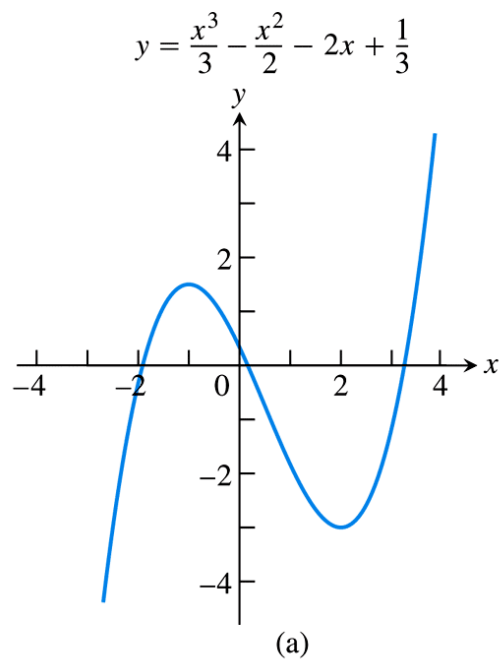


FIGURE 1.18 Graphs of three polynomial functions.

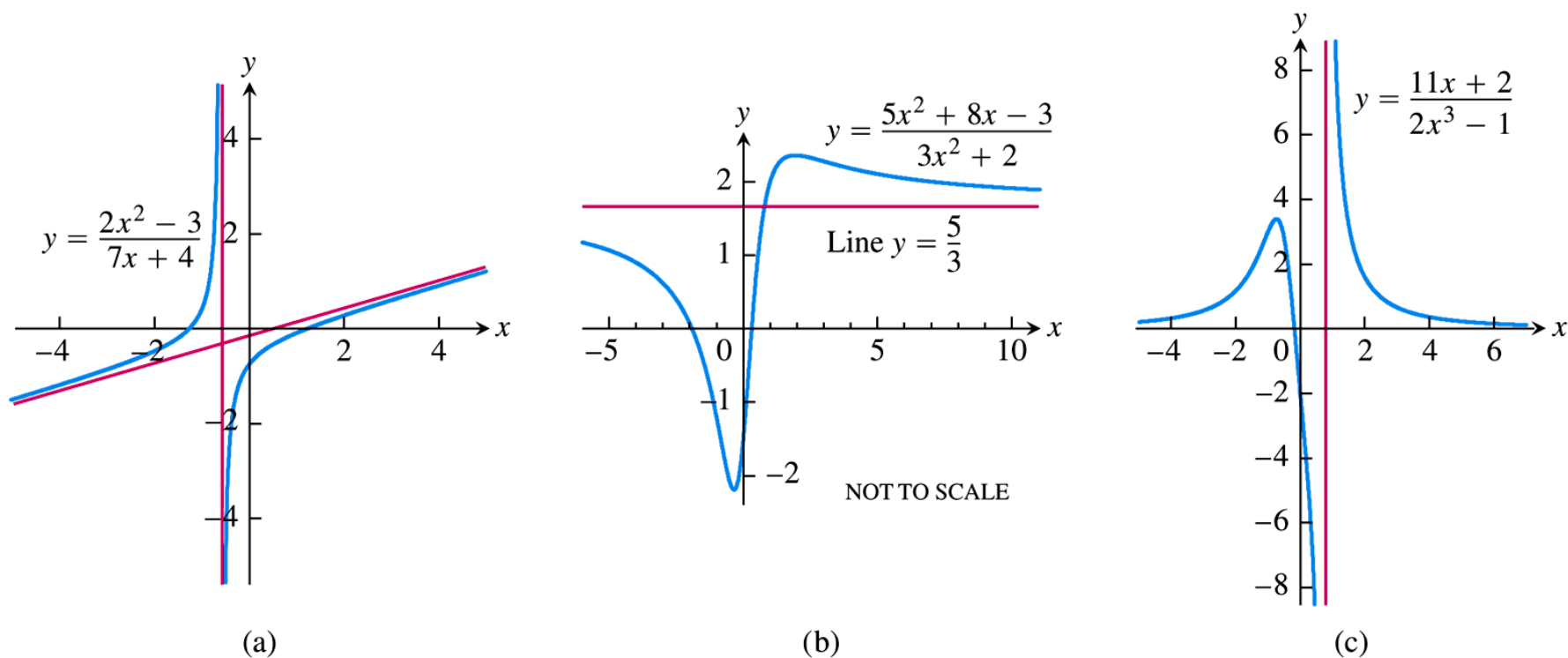
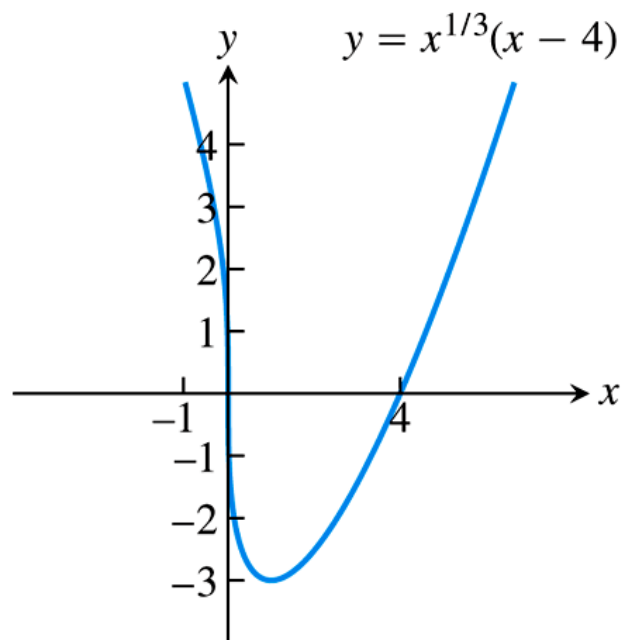
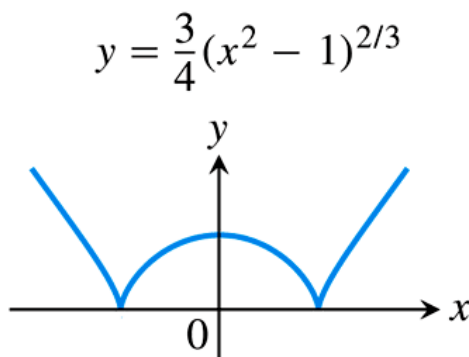


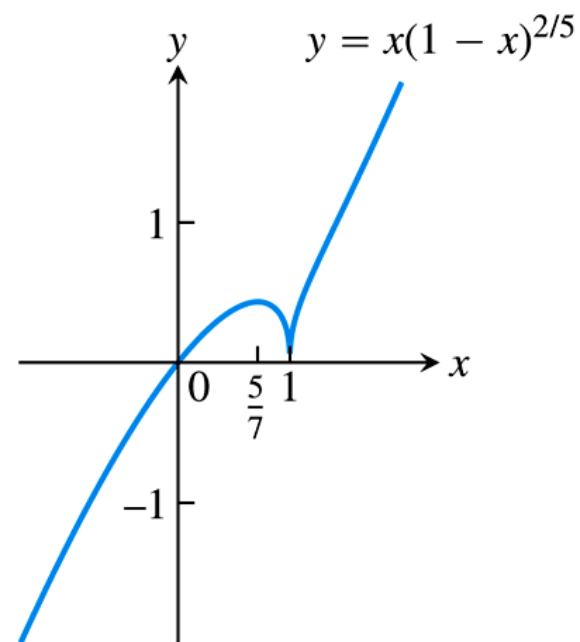
FIGURE 1.19 Graphs of three rational functions. The straight red lines are called *asymptotes* and are not part of the graph.



(a)

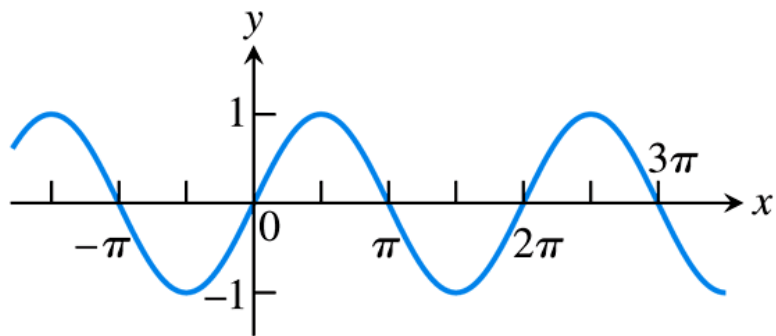


(b)

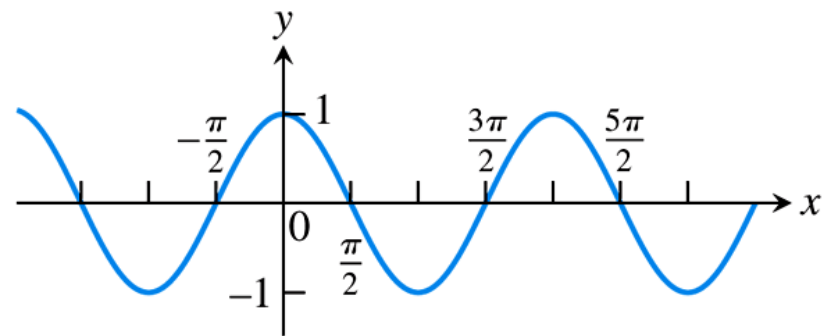


(c)

FIGURE 1.20 Graphs of three algebraic functions.



(a) $f(x) = \sin x$



(b) $f(x) = \cos x$

FIGURE 1.21 Graphs of the sine and cosine functions.

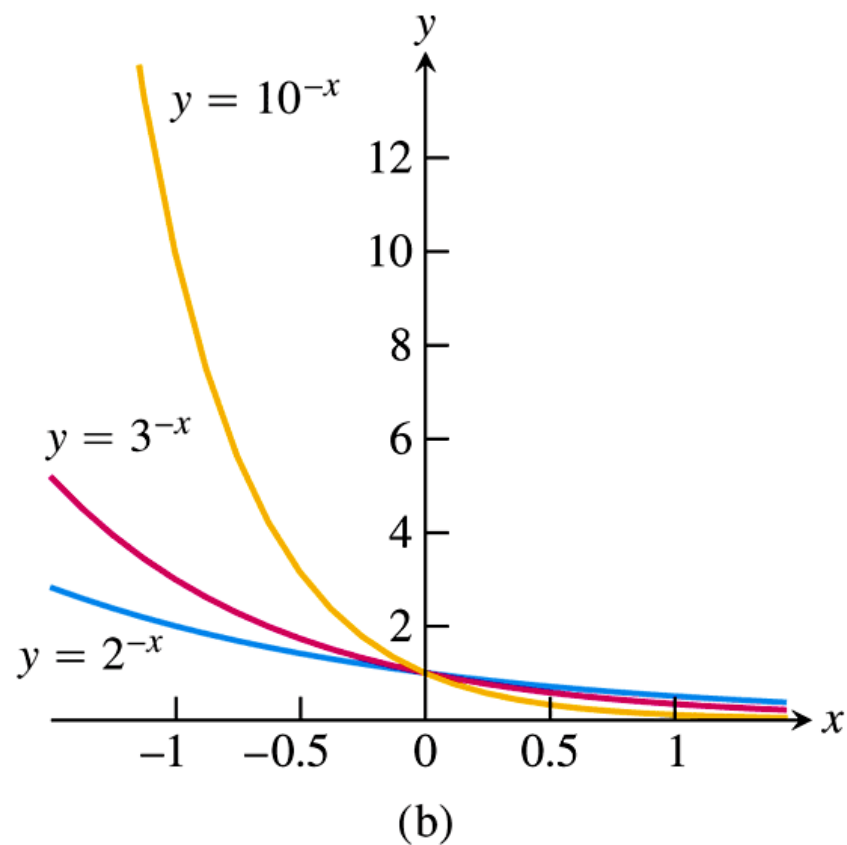
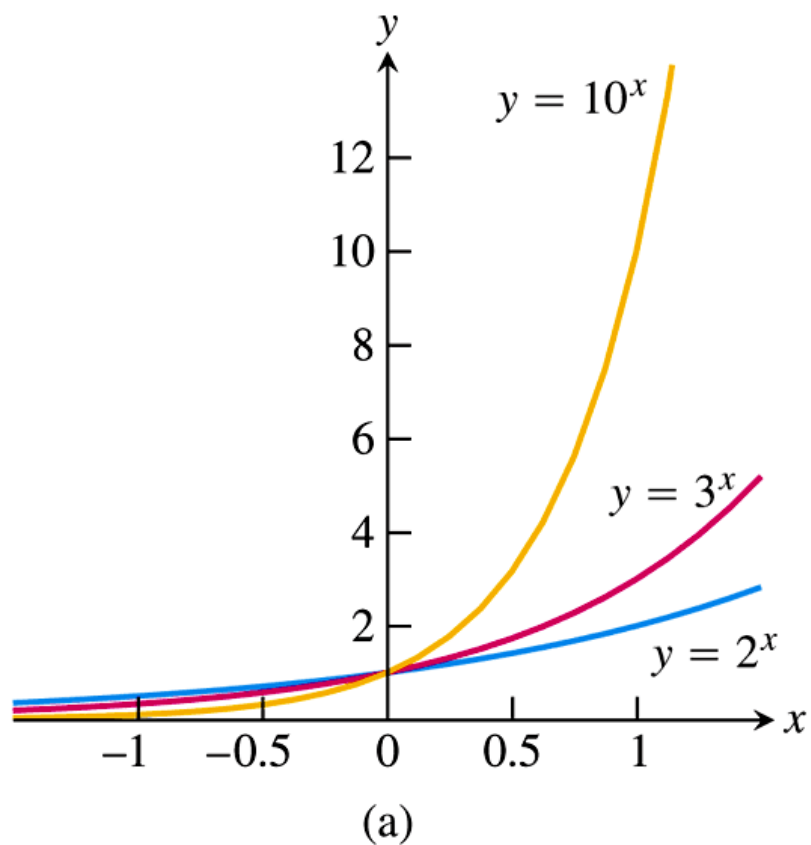


FIGURE 1.22 Graphs of exponential functions.

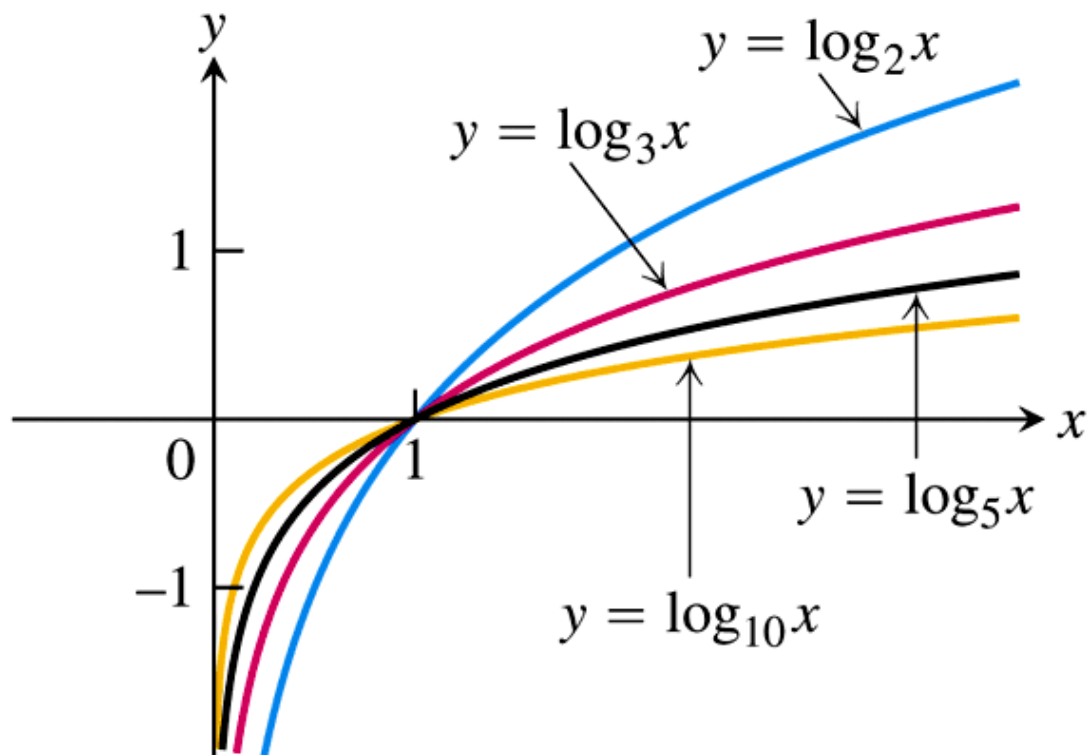


FIGURE 1.23 Graphs of four logarithmic functions.

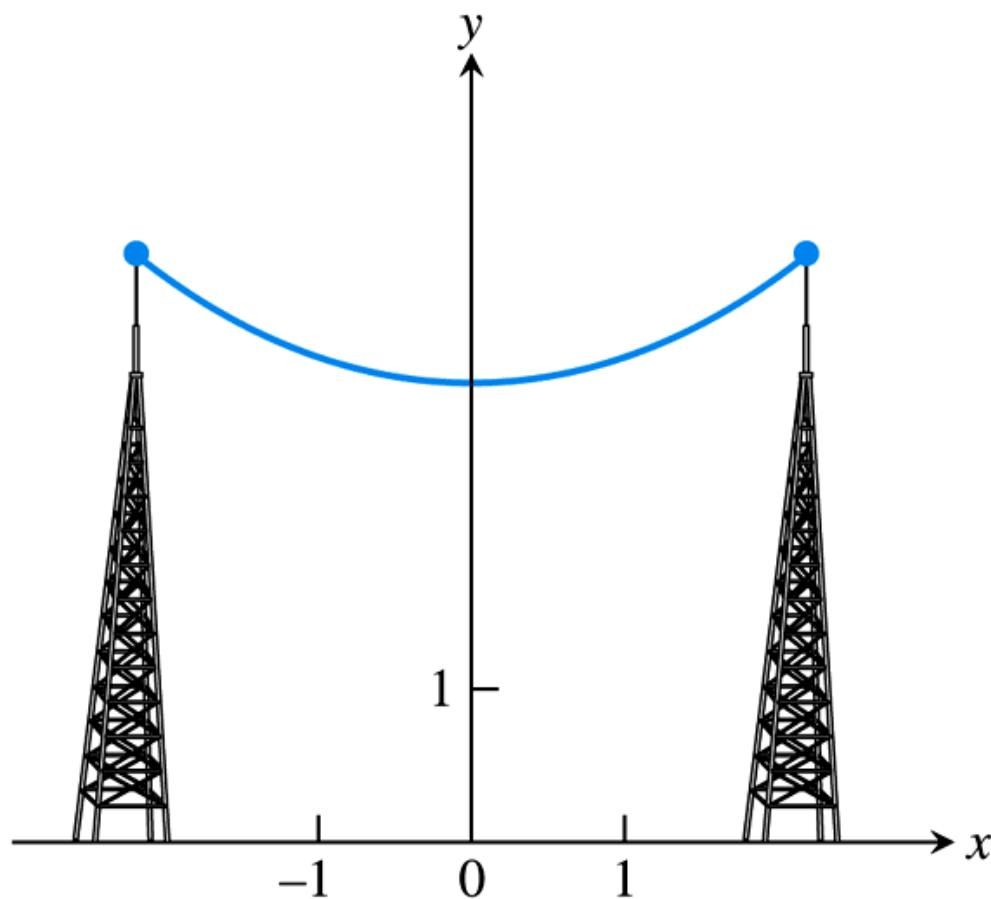


FIGURE 1.24 Graph of a catenary or hanging cable. (The Latin word *catena* means “chain.”)

1.2

Combining Functions; Shifting and Scaling Graphs

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

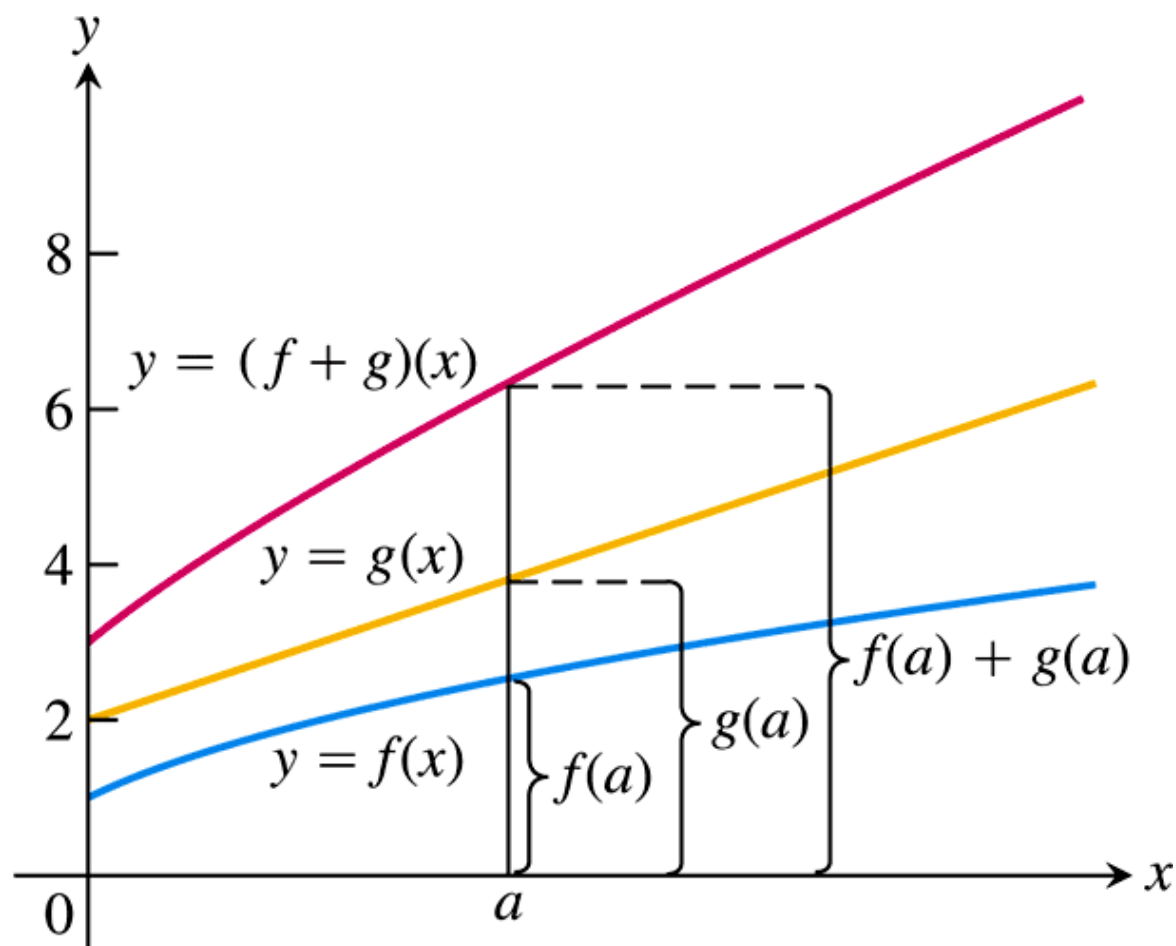


FIGURE 1.25 Graphical addition of two functions.

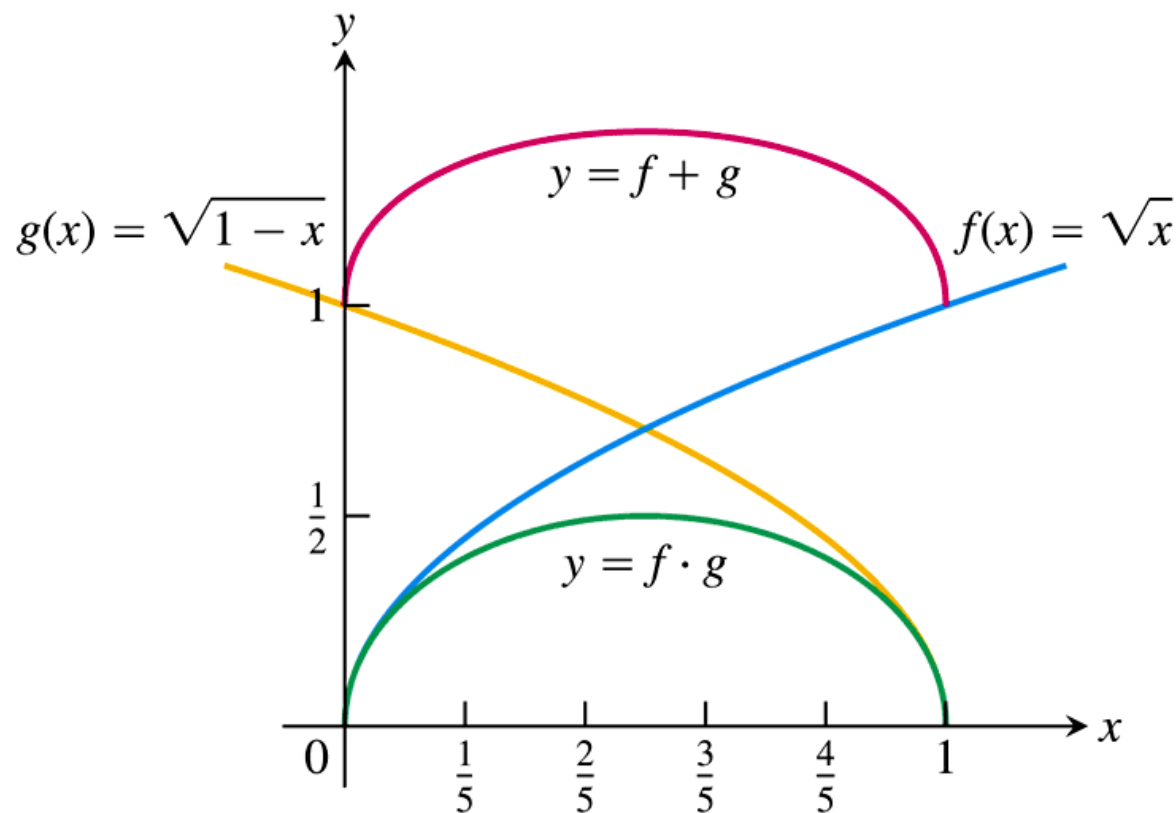


FIGURE 1.26 The domain of the function $f + g$ is the intersection of the domains of f and g , the interval $[0, 1]$ on the x -axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

DEFINITION If f and g are functions, the **composite** function $f \circ g$ (“ f composed with g ”) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f .

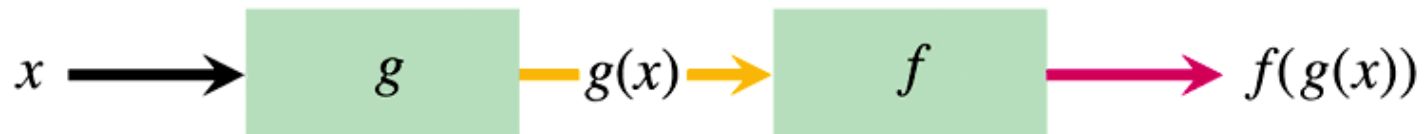


FIGURE 1.27 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

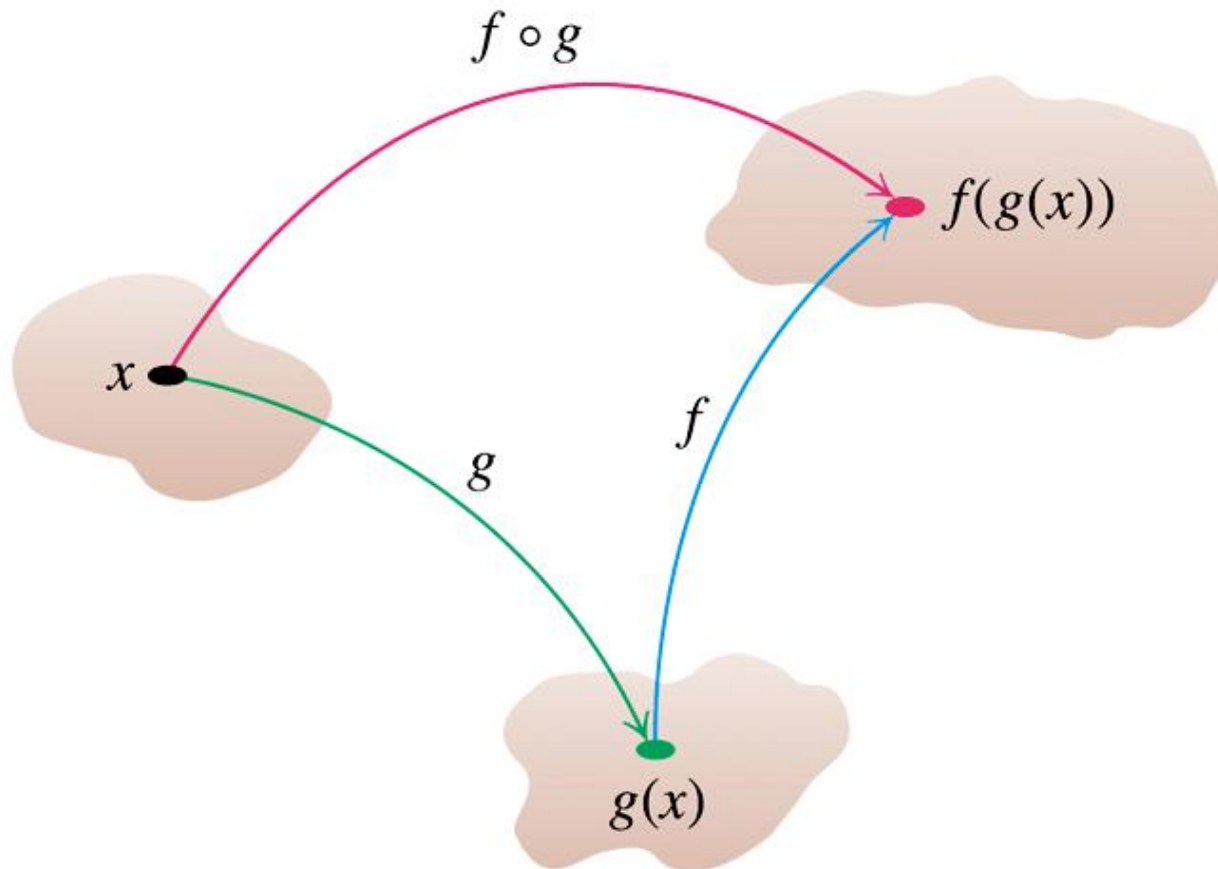


FIGURE 1.28 Arrow diagram for $f \circ g$.

Shift Formulas

Vertical Shifts

$$y = f(x) + k$$

Shifts the graph of f *up* k units if $k > 0$

Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$$y = f(x + h)$$

Shifts the graph of f *left* h units if $h > 0$

Shifts it *right* $|h|$ units if $h < 0$

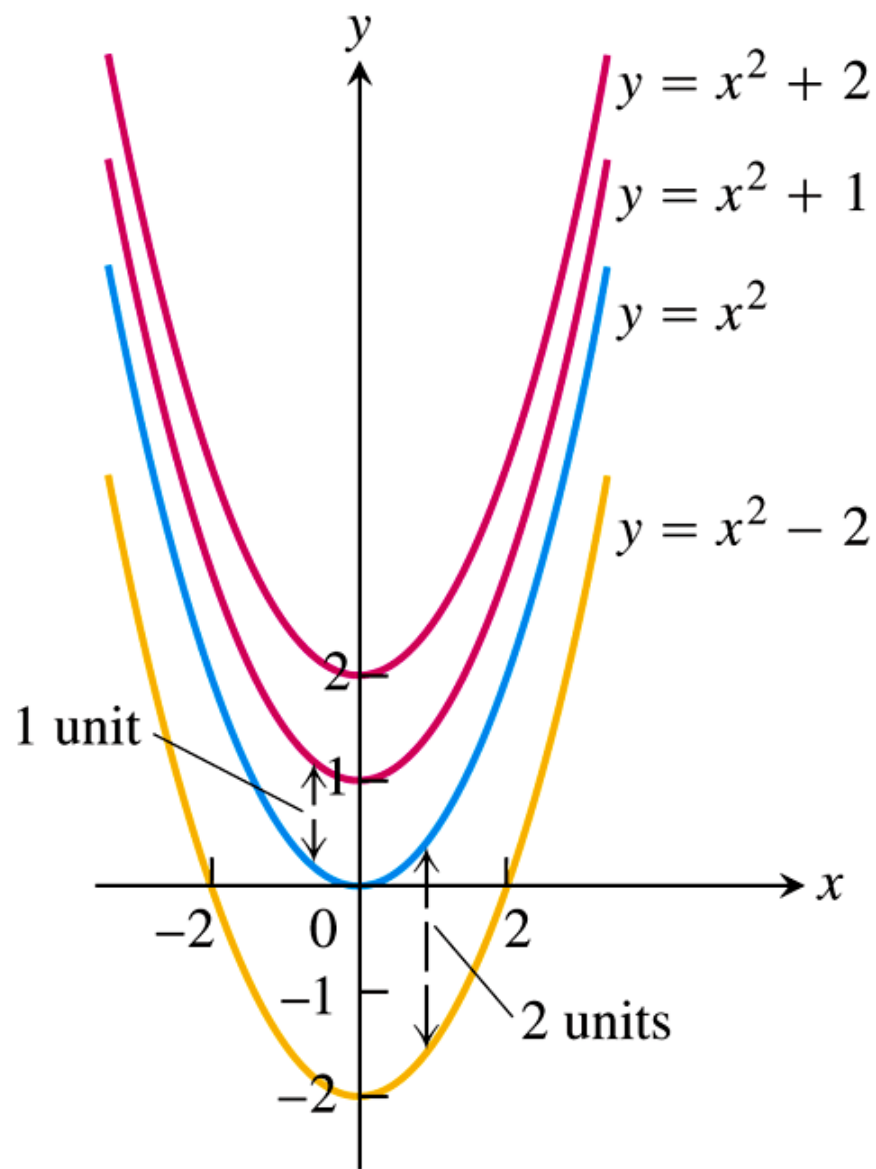


FIGURE 1.29 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Examples 3a and b).

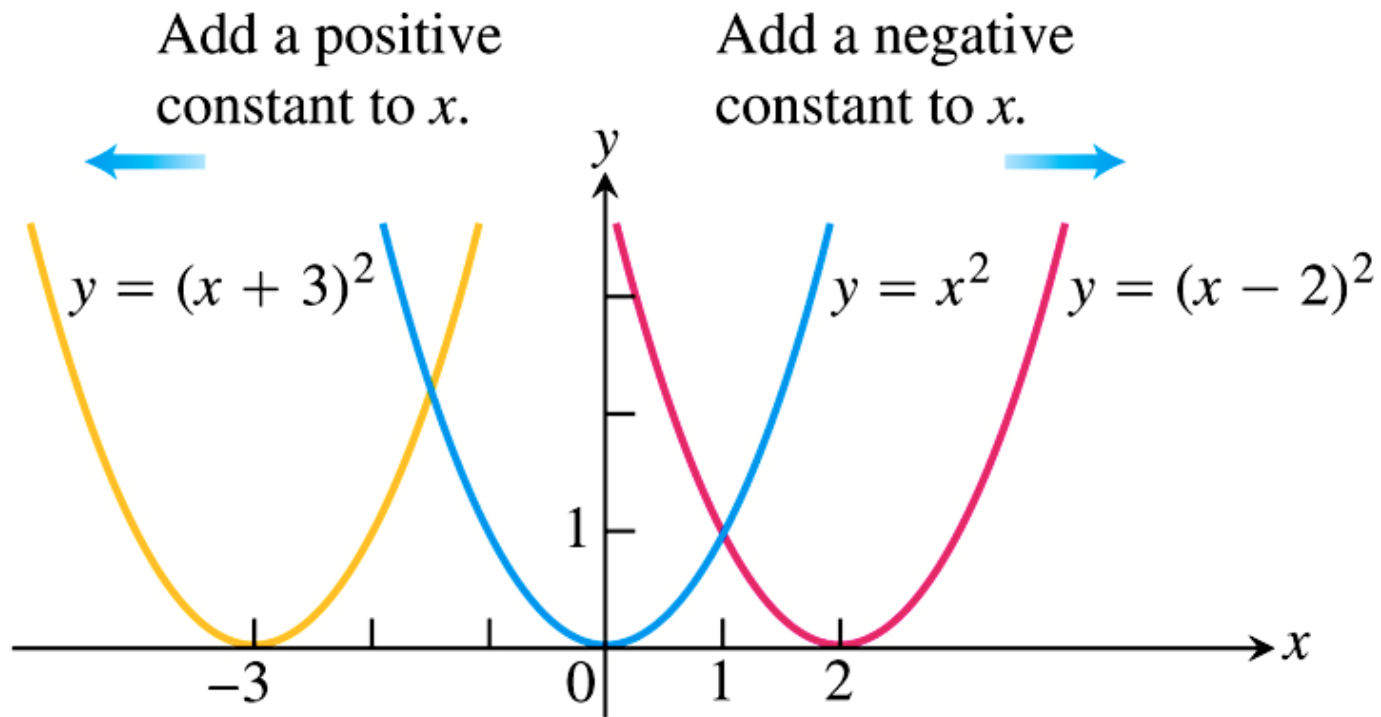


FIGURE 1.30 To shift the graph of $y = x^2$ to the left, we add a positive constant to x (Example 3c). To shift the graph to the right, we add a negative constant to x .

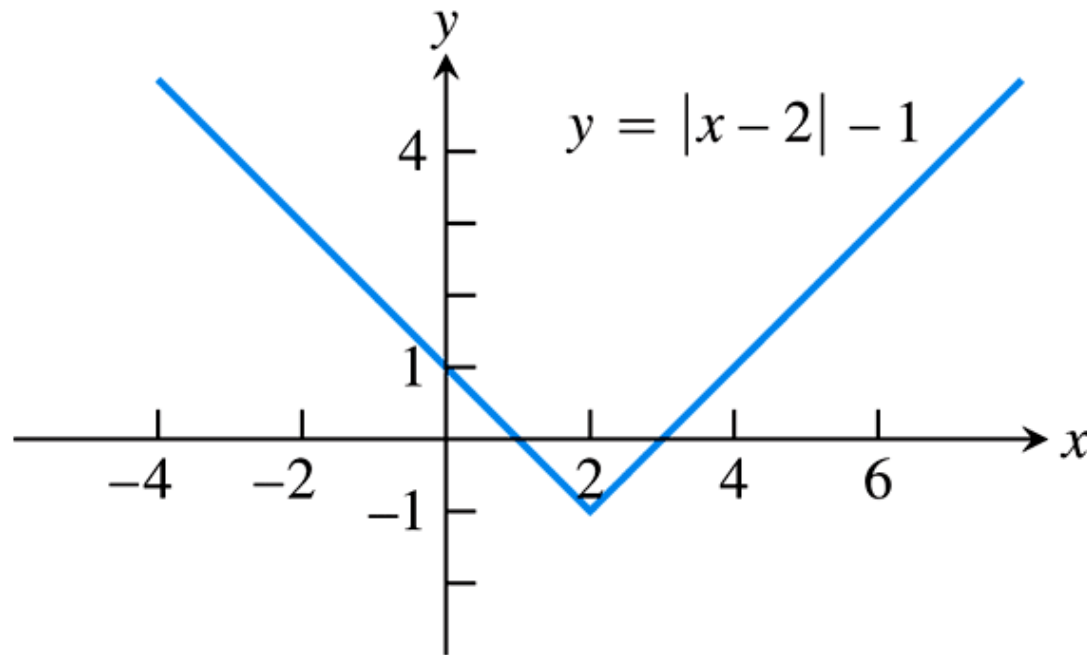


FIGURE 1.31 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 3d).

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$ Stretches the graph of f vertically by a factor of c .

$y = \frac{1}{c}f(x)$ Compresses the graph of f vertically by a factor of c .

$y = f(cx)$ Compresses the graph of f horizontally by a factor of c .

$y = f(x/c)$ Stretches the graph of f horizontally by a factor of c .

For $c = -1$, the graph is reflected:

$y = -f(x)$ Reflects the graph of f across the x -axis.

$y = f(-x)$ Reflects the graph of f across the y -axis.

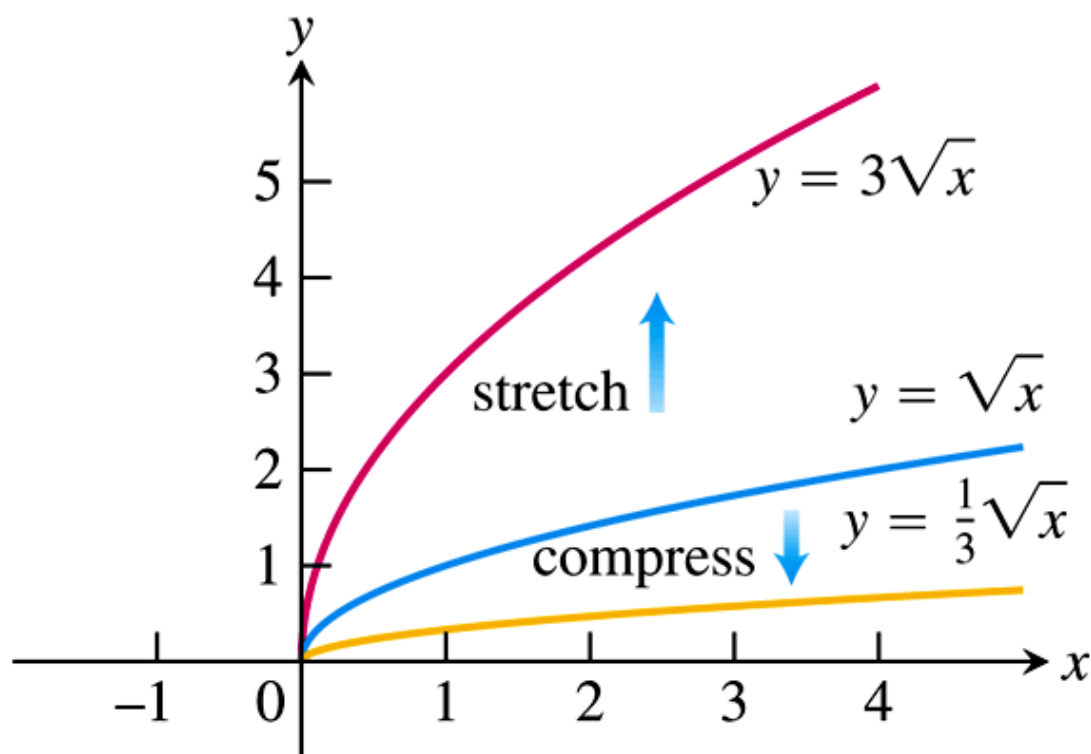


FIGURE 1.32 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4a).

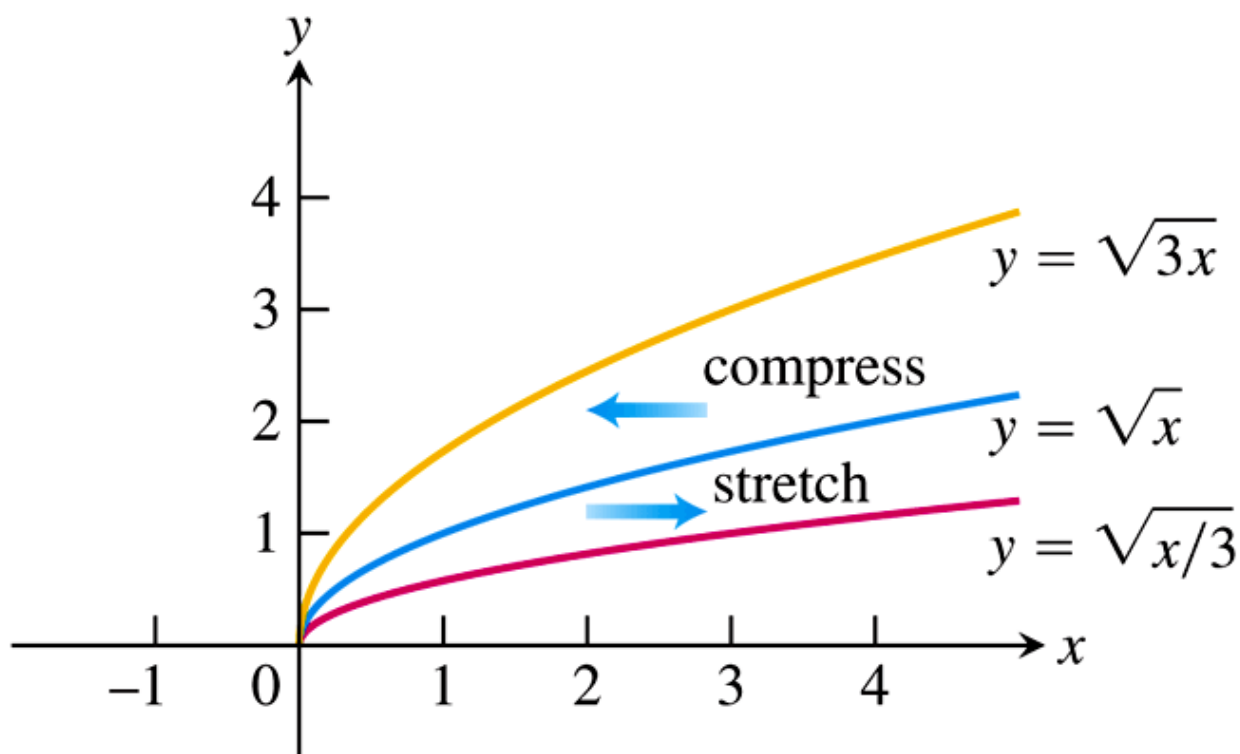


FIGURE 1.33 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4b).

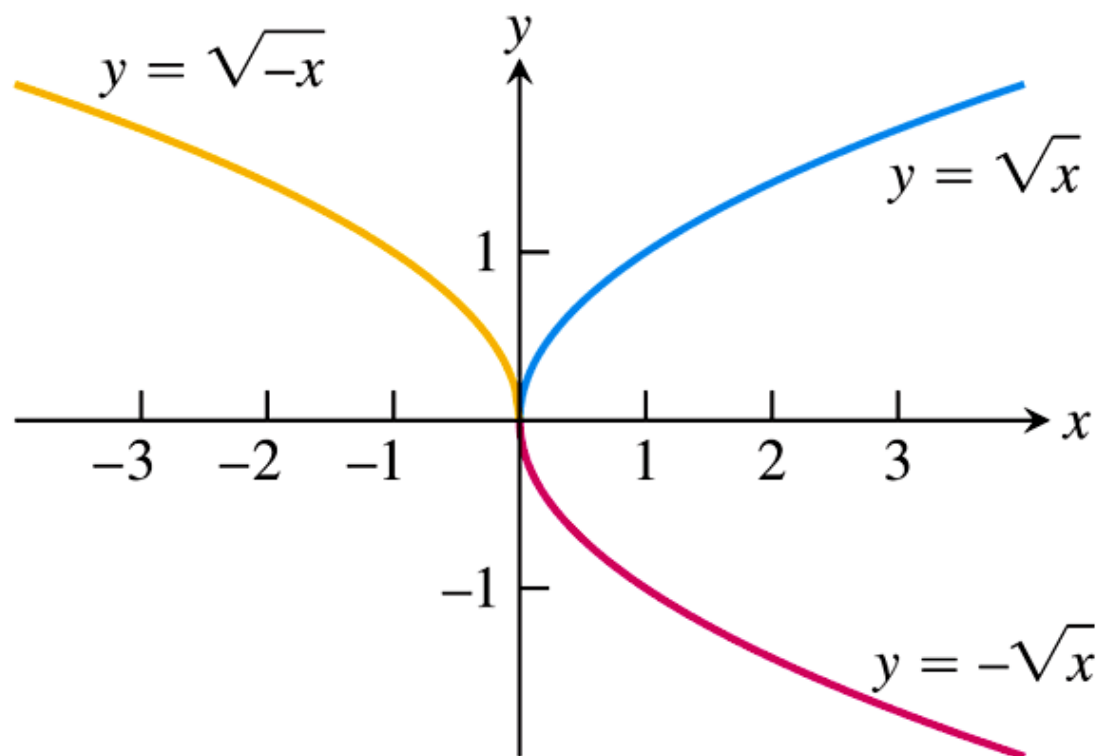
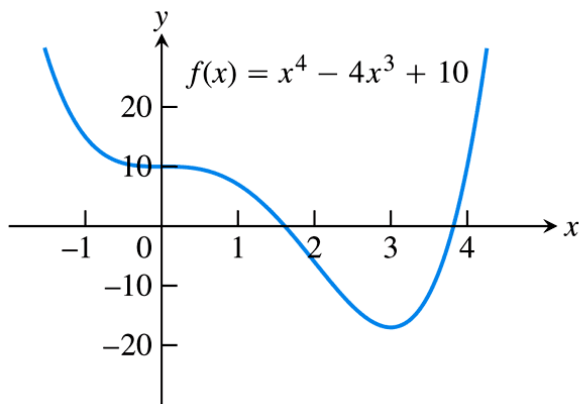
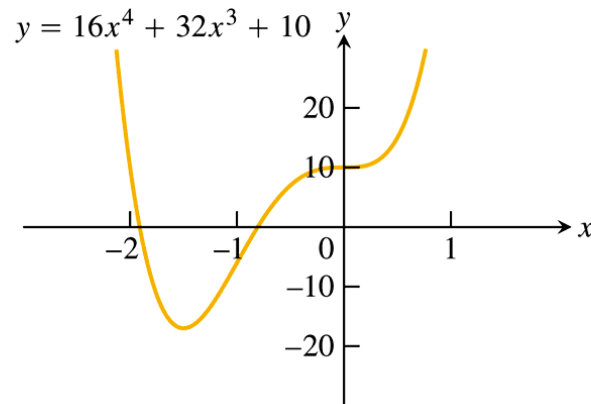


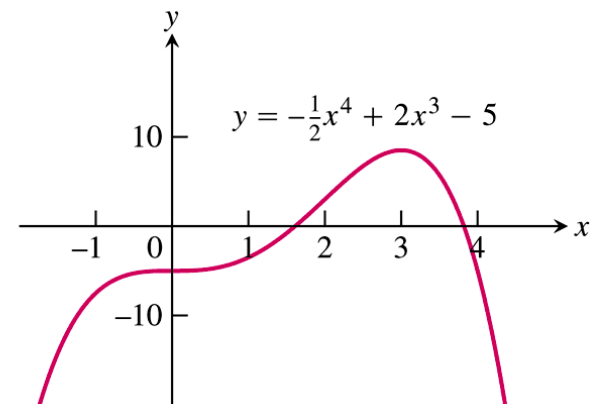
FIGURE 1.34 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 4c).



(a)

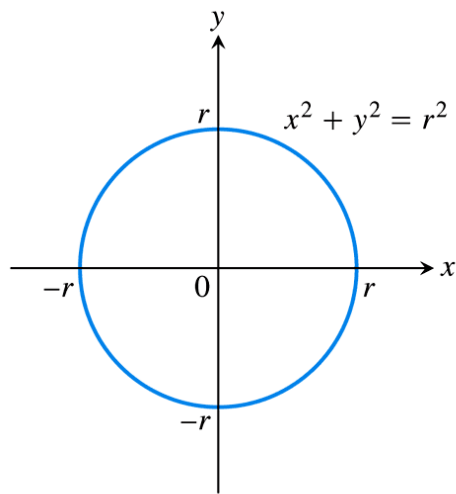


(b)

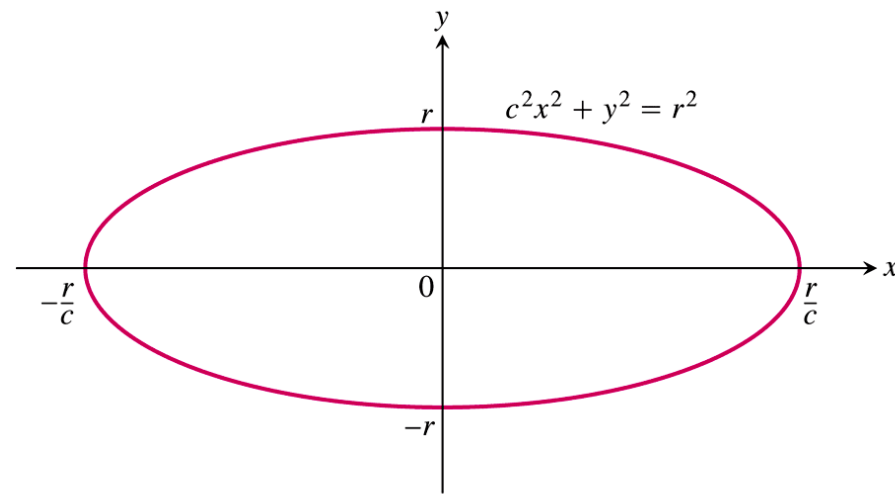


(c)

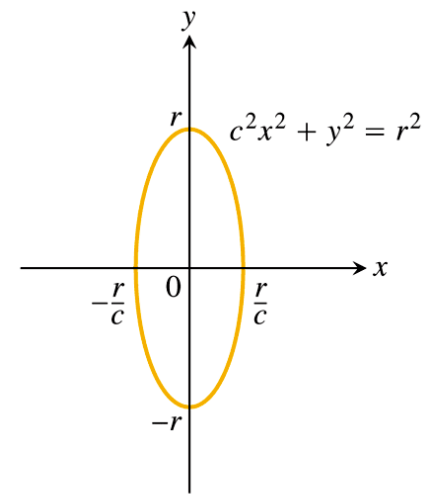
FIGURE 1.35 (a) The original graph of f . (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y -axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x -axis (Example 5).



(a) circle



(b) ellipse, $0 < c < 1$



(c) ellipse, $c > 1$

FIGURE 1.36 Horizontal stretching or compression of a circle produces graphs of ellipses.

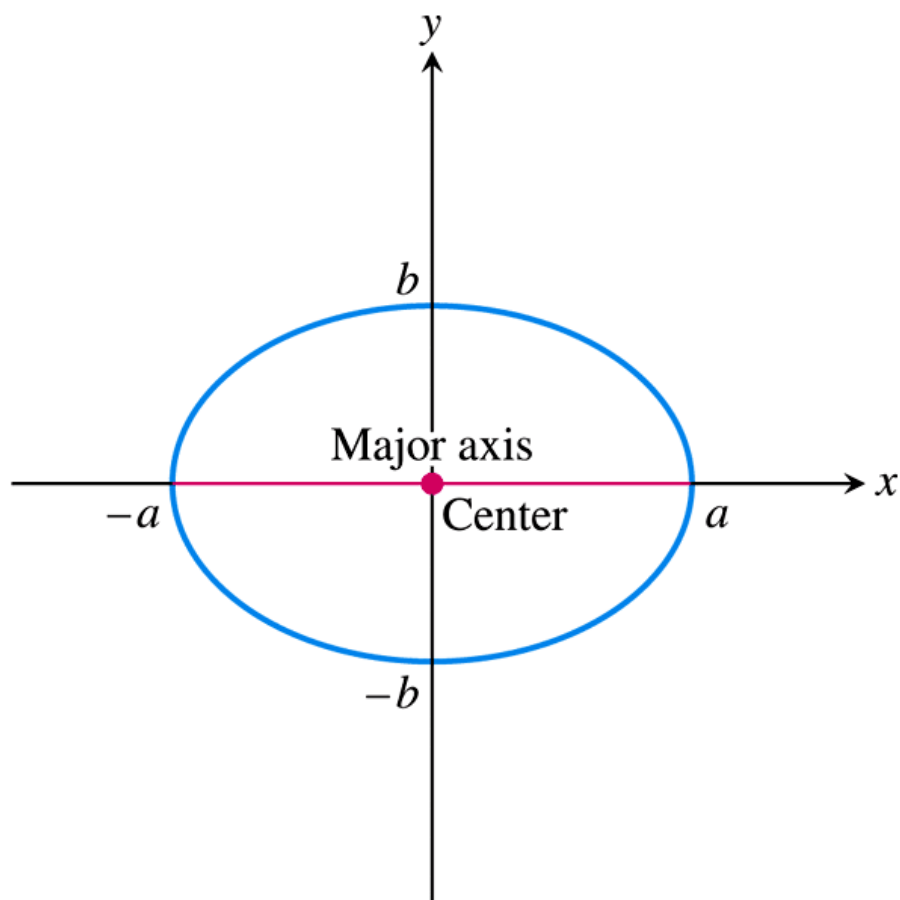


FIGURE 1.37 Graph of the ellipse
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$, where the major
axis is horizontal.

1.3

Trigonometric Functions

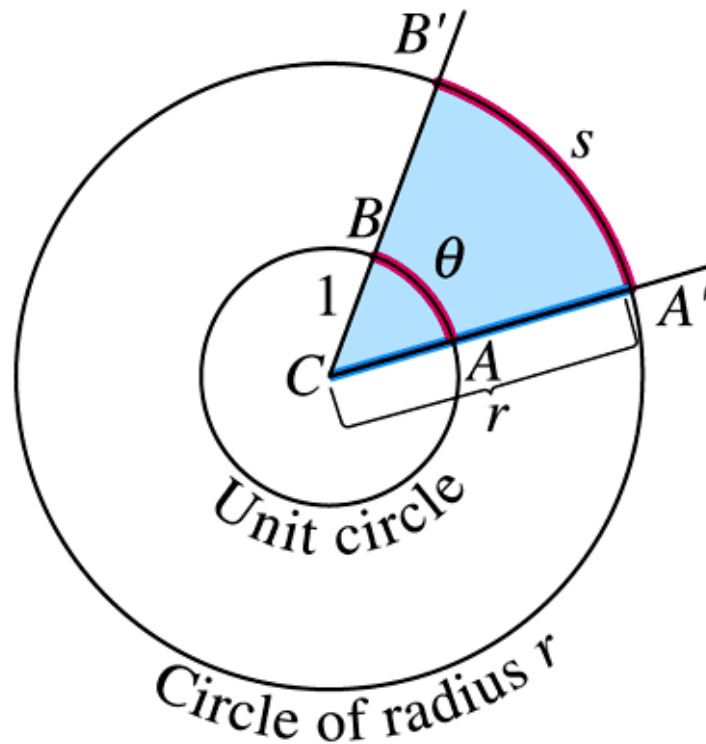


FIGURE 1.38 The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

$$s = r\theta \quad (\theta \text{ in radians}). \quad (1)$$

TABLE 1.2 Angles measured in degrees and radians

Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

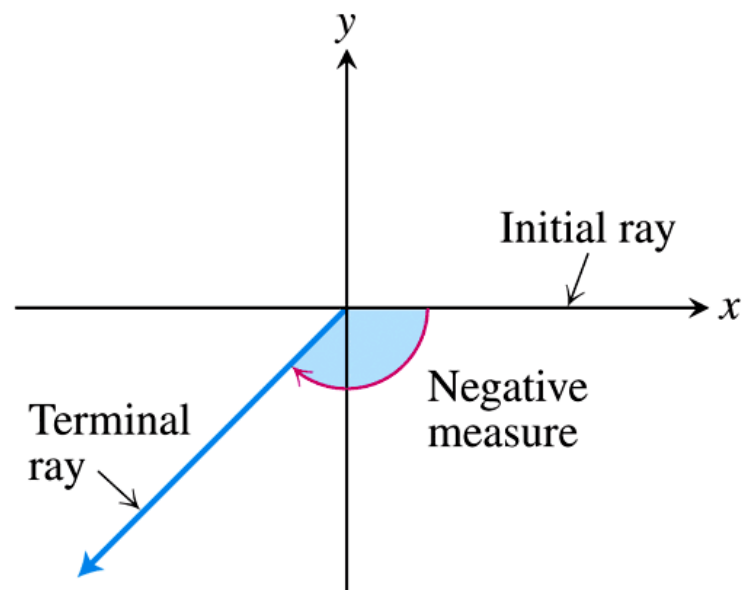
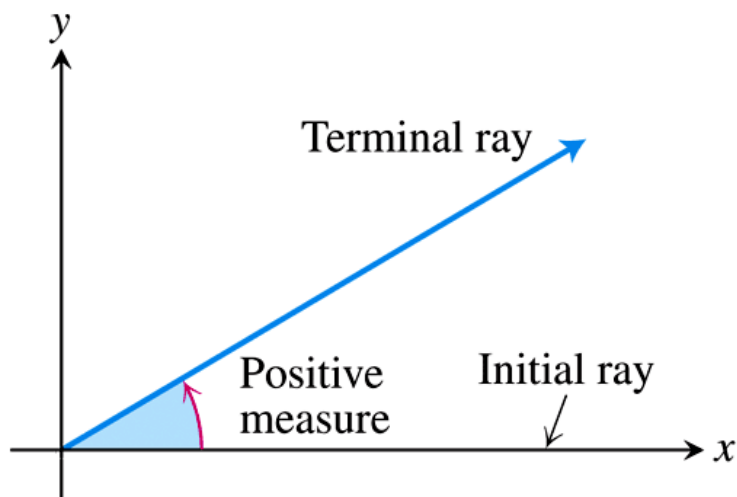


FIGURE 1.39 Angles in standard position in the xy -plane.

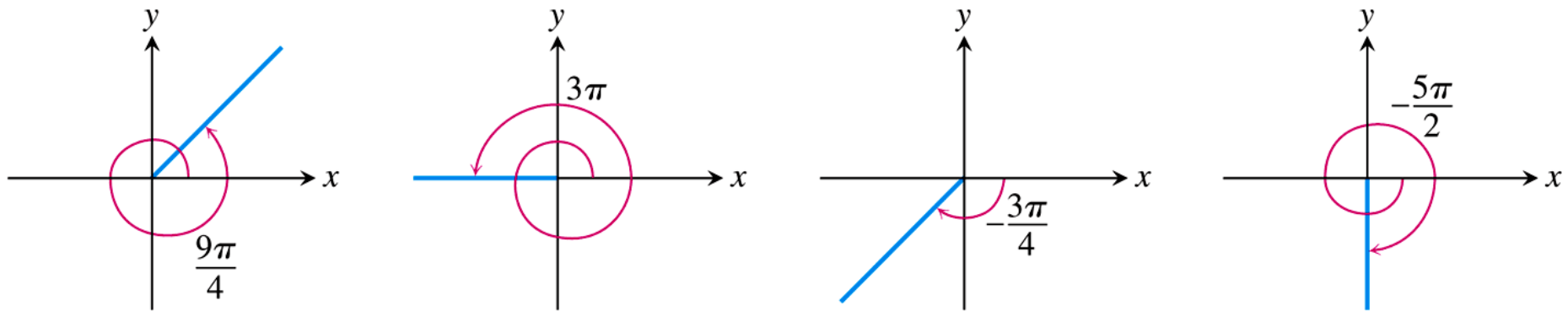
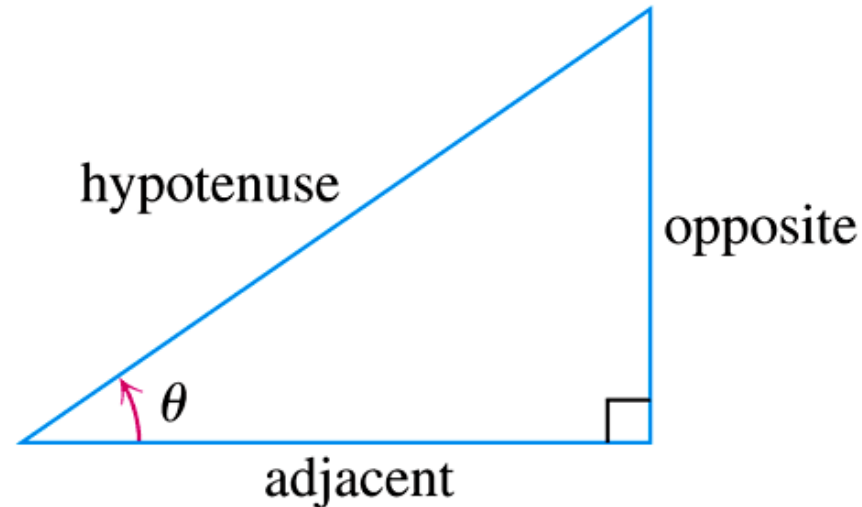


FIGURE 1.40 Nonzero radian measures can be positive or negative and can go beyond 2π .



$$\begin{aligned}\sin \theta &= \frac{\text{opp}}{\text{hyp}} & \csc \theta &= \frac{\text{hyp}}{\text{opp}} \\ \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} & \cot \theta &= \frac{\text{adj}}{\text{opp}}\end{aligned}$$

FIGURE 1.41 Trigonometric ratios of an acute angle.

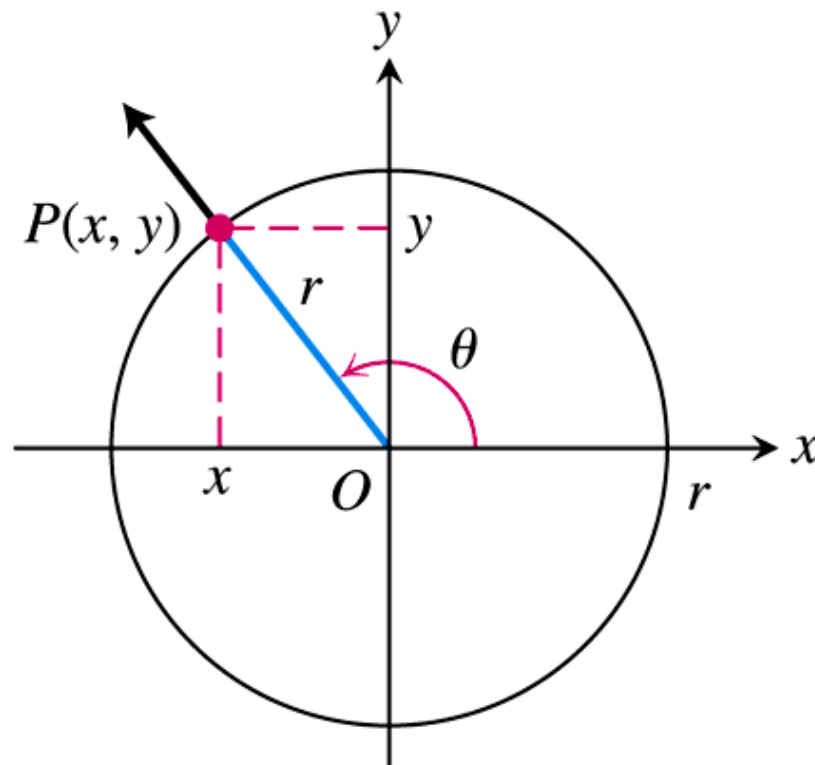


FIGURE 1.42 The trigonometric functions of a general angle θ are defined in terms of x , y , and r .

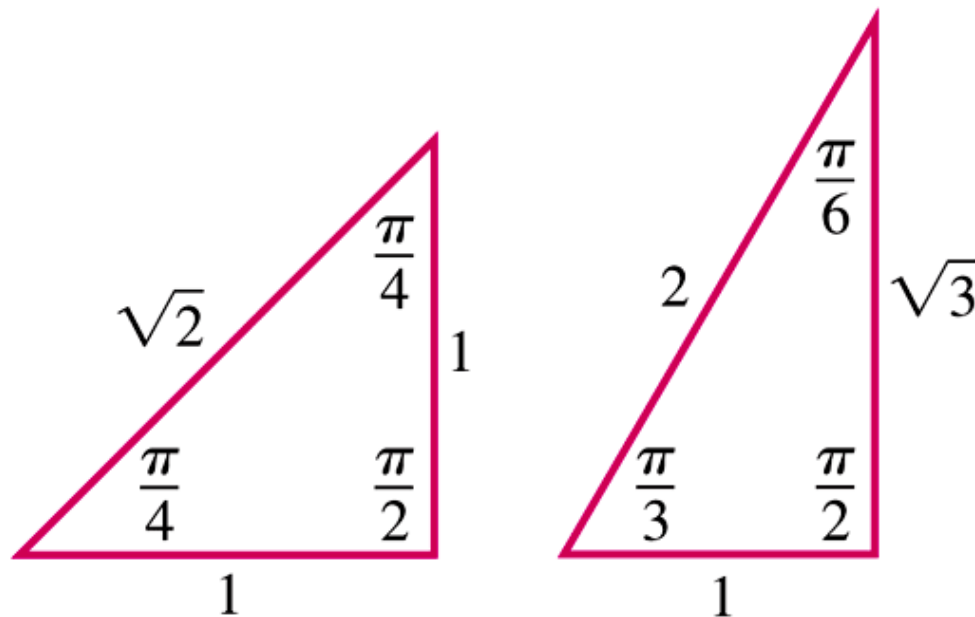


FIGURE 1.43 Radian angles and side lengths of two common triangles.

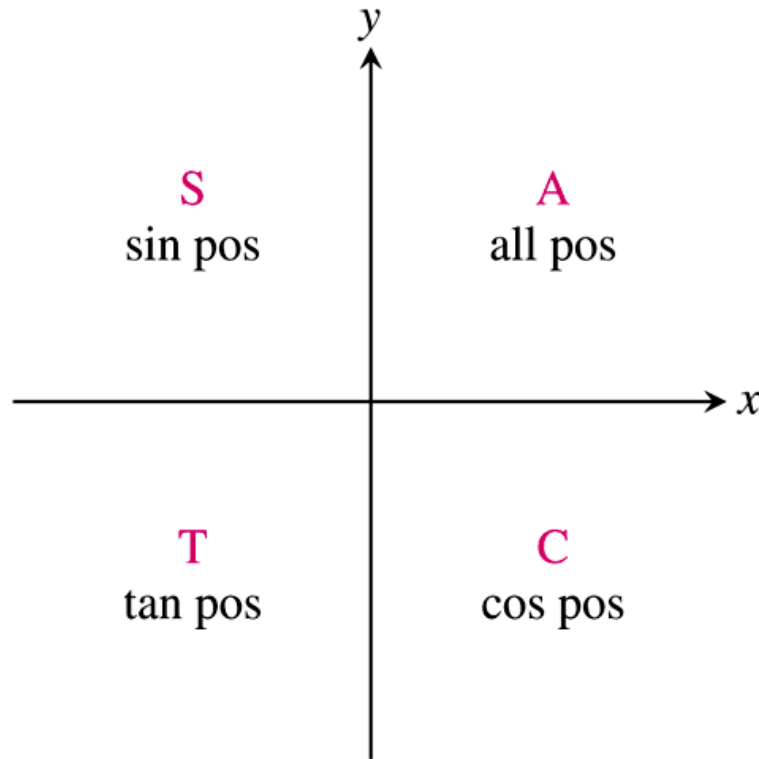


FIGURE 1.44 The CAST rule, remembered by the statement “Calculus Activates Student Thinking,” tells which trigonometric functions are positive in each quadrant.

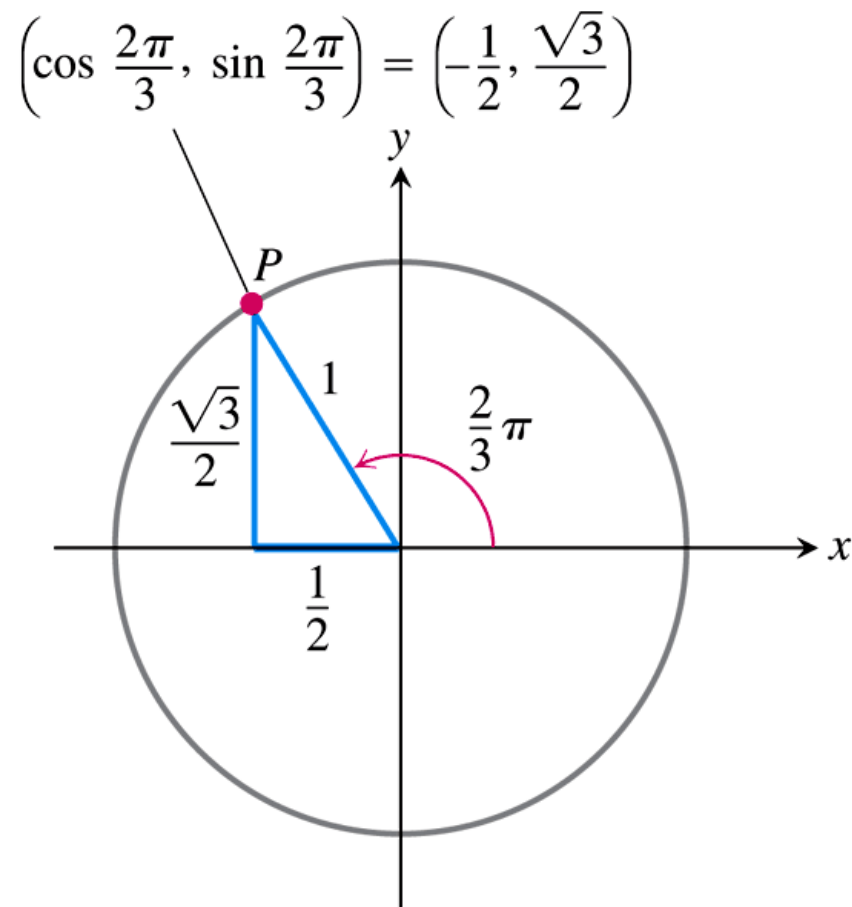


FIGURE 1.45 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

TABLE 1.3 Values of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for selected values of θ

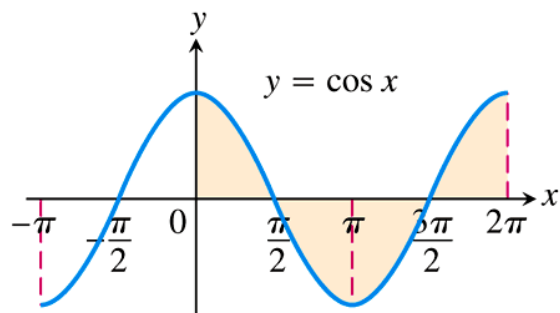
Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos \theta$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
$\tan \theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0		0

Periods of Trigonometric Functions

Period π : $\tan(x + \pi) = \tan x$
 $\cot(x + \pi) = \cot x$

Period 2π : $\sin(x + 2\pi) = \sin x$
 $\cos(x + 2\pi) = \cos x$
 $\sec(x + 2\pi) = \sec x$
 $\csc(x + 2\pi) = \csc x$

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x . The smallest such value of p is the **period** of f .

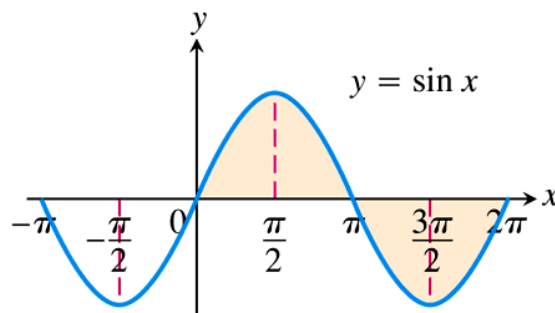


Domain: $-\infty < x < \infty$

Range: $-1 \leq y \leq 1$

Period: 2π

(a)

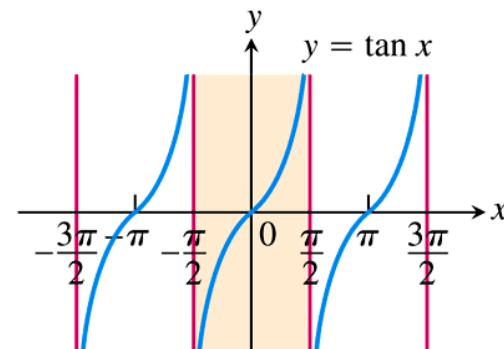


Domain: $-\infty < x < \infty$

Range: $-1 \leq y \leq 1$

Period: 2π

(b)

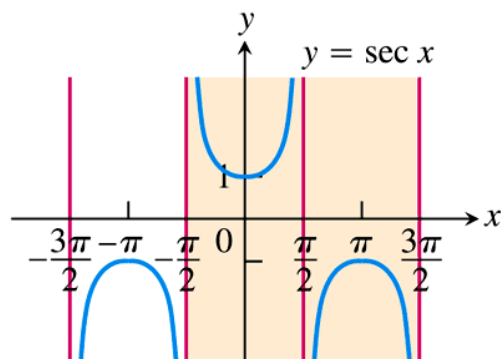


Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$

Range: $-\infty < y < \infty$

Period: π

(c)

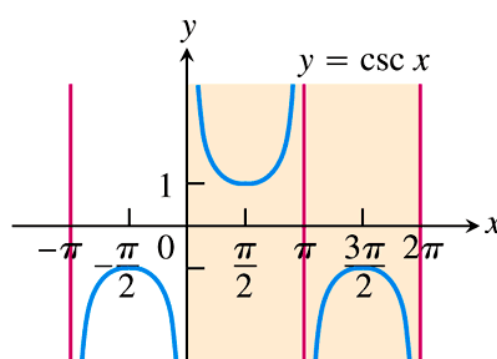


Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$

Range: $y \leq -1$ or $y \geq 1$

Period: 2π

(d)

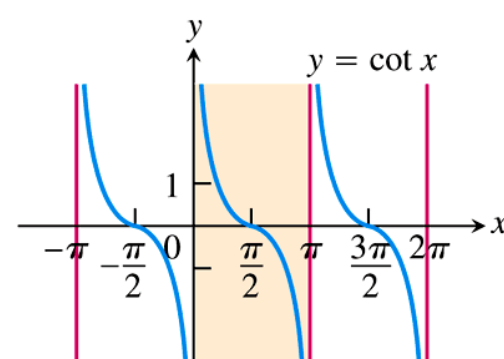


Domain: $x \neq 0, \pm\pi, \pm2\pi, \dots$

Range: $y \leq -1$ or $y \geq 1$

Period: 2π

(e)



Domain: $x \neq 0, \pm\pi, \pm2\pi, \dots$

Range: $-\infty < y < \infty$

Period: π

(f)

FIGURE 1.46 Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Even

$$\cos(-x) = \cos x$$

$$\sec(-x) = \sec x$$

Odd

$$\sin(-x) = -\sin x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\cot(-x) = -\cot x$$

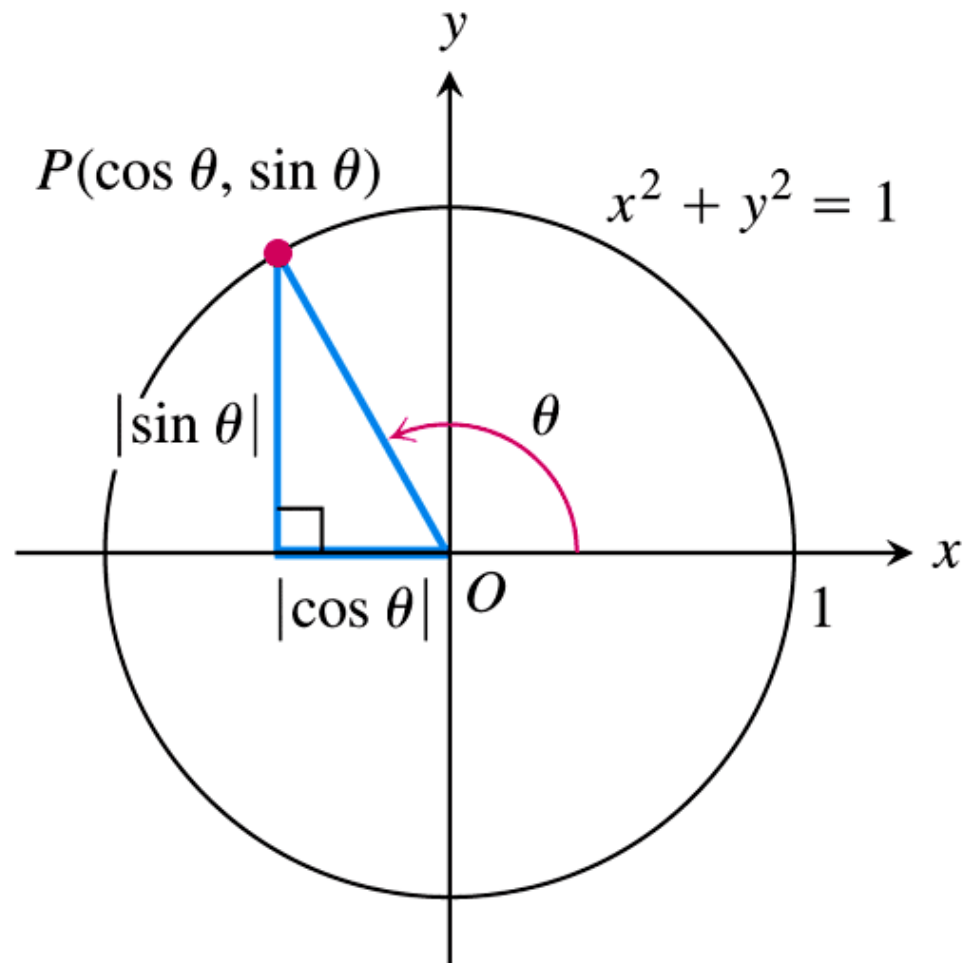


FIGURE 1.47 The reference triangle for a general angle θ .

$$\cos^2 \theta + \sin^2 \theta = 1. \quad (3)$$

$$1 + \tan^2 \theta = \sec^2 \theta.$$

$$1 + \cot^2 \theta = \csc^2 \theta.$$

Addition Formulas

$$\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B \end{aligned} \quad (4)$$

Double-Angle Formulas

$$\begin{aligned}\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ \sin 2\theta &= 2 \sin \theta \cos \theta\end{aligned}\tag{5}$$

Half-Angle Formulas

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}\tag{6}$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}\tag{7}$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta.\tag{8}$$

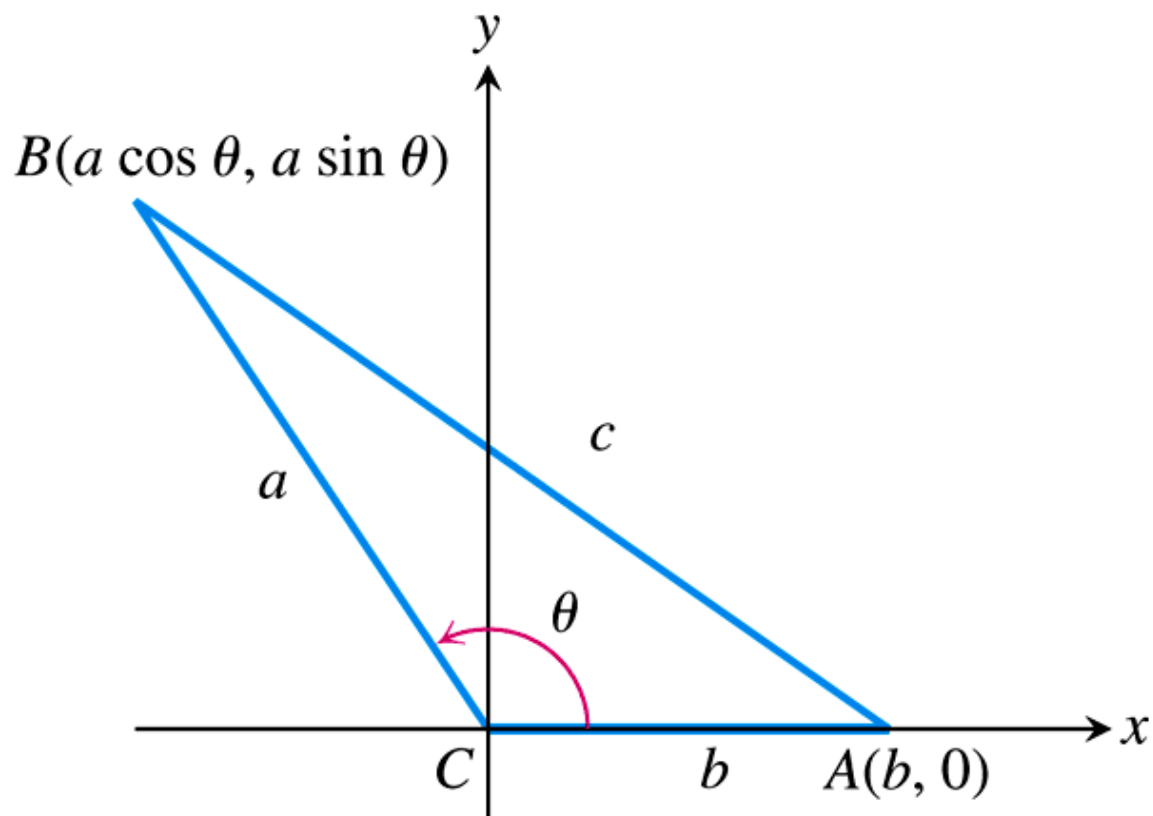


FIGURE 1.48 The square of the distance between A and B gives the law of cosines.

Vertical stretch or compression;
reflection about x -axis if negative

$$y = af(b(x + c)) + d$$

Vertical shift

Horizontal stretch or compression;
reflection about y -axis if negative

Horizontal shift

Two Special Inequalities

For any angle θ measured in radians,

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta|.$$

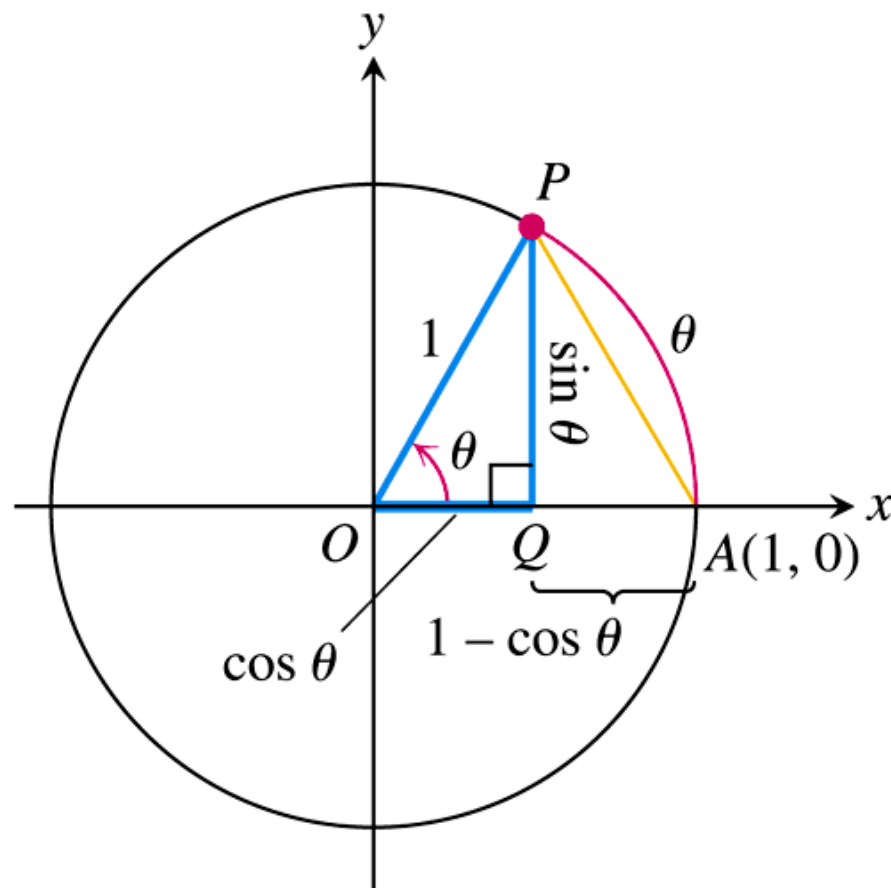


FIGURE 1.49 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2 \theta + (1 - \cos \theta)^2 \leq \theta^2$.

1.4

Graphing with Calculators and Computers

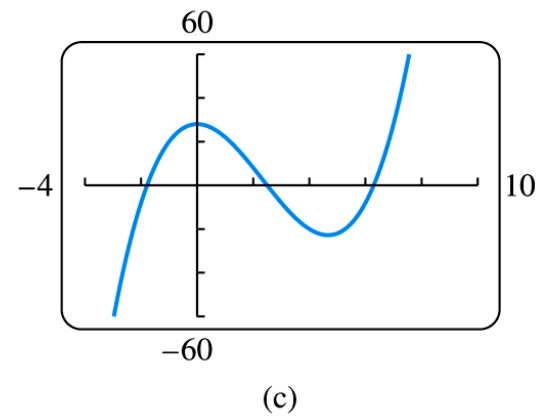
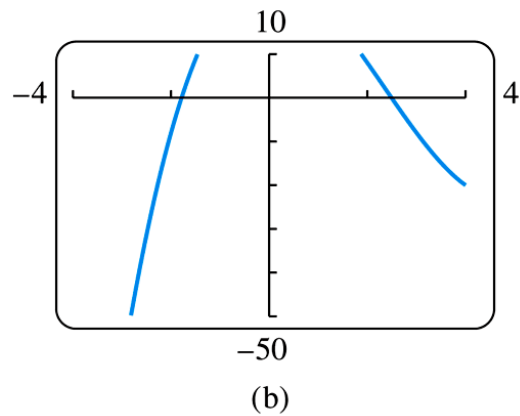
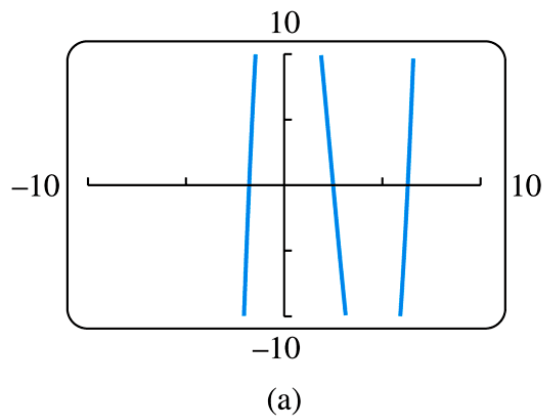


FIGURE 1.50 The graph of $f(x) = x^3 - 7x^2 + 28$ in different viewing windows. Selecting a window that gives a clear picture of a graph is often a trial-and-error process (Example 1).

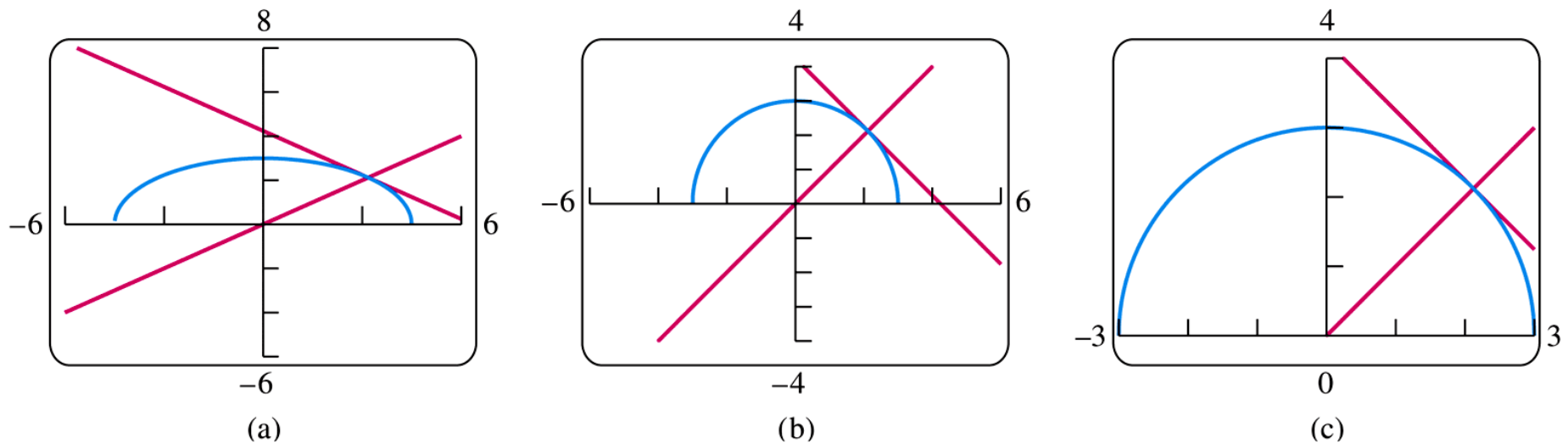


FIGURE 1.51 Graphs of the perpendicular lines $y = x$ and $y = -x + 3\sqrt{2}$, and the semicircle $y = \sqrt{9 - x^2}$ appear distorted (a) in a nonsquare window, but clear (b) and (c) in square windows (Example 2).

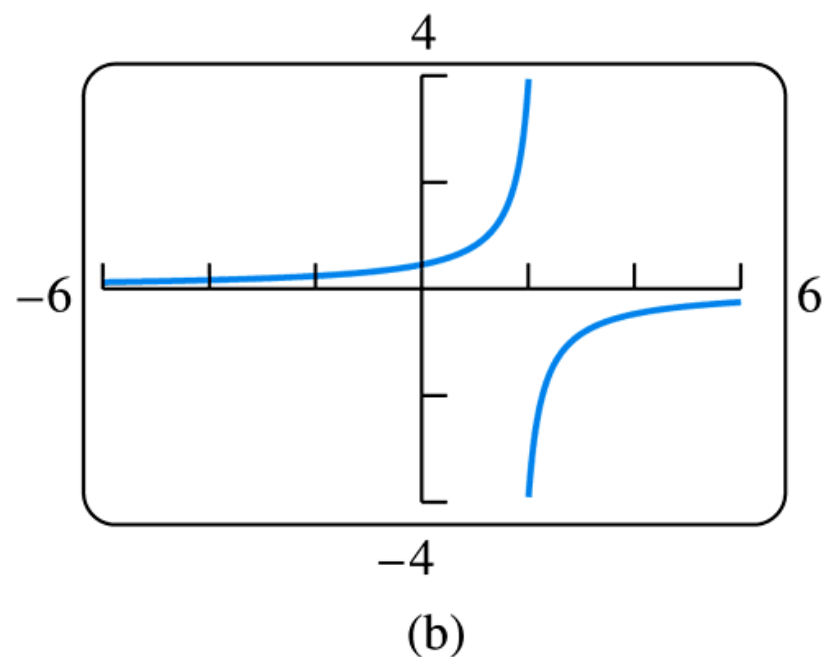
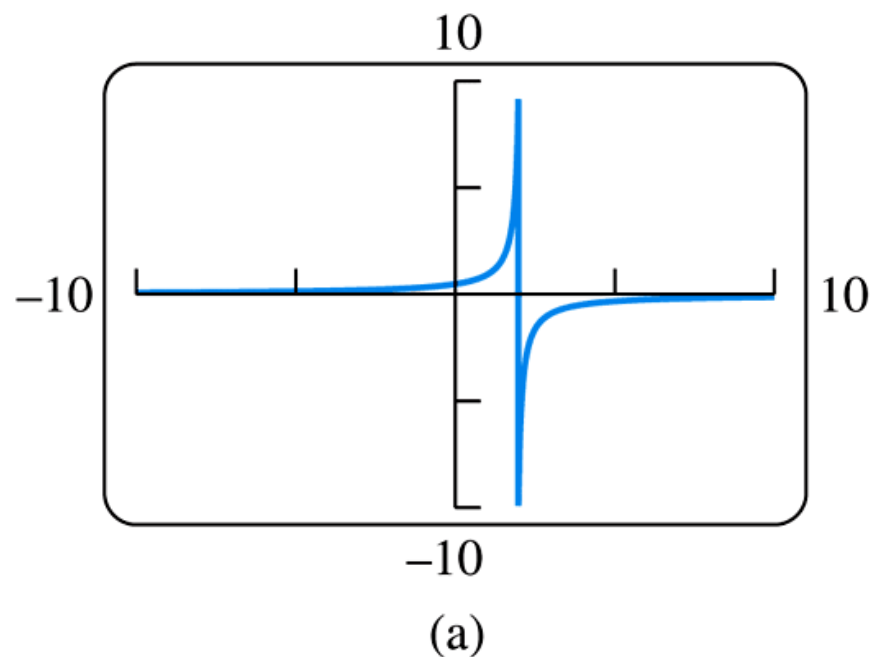


FIGURE 1.52 Graphs of the function $y = \frac{1}{2-x}$. A vertical line may appear without a careful choice of the viewing window (Example 3).

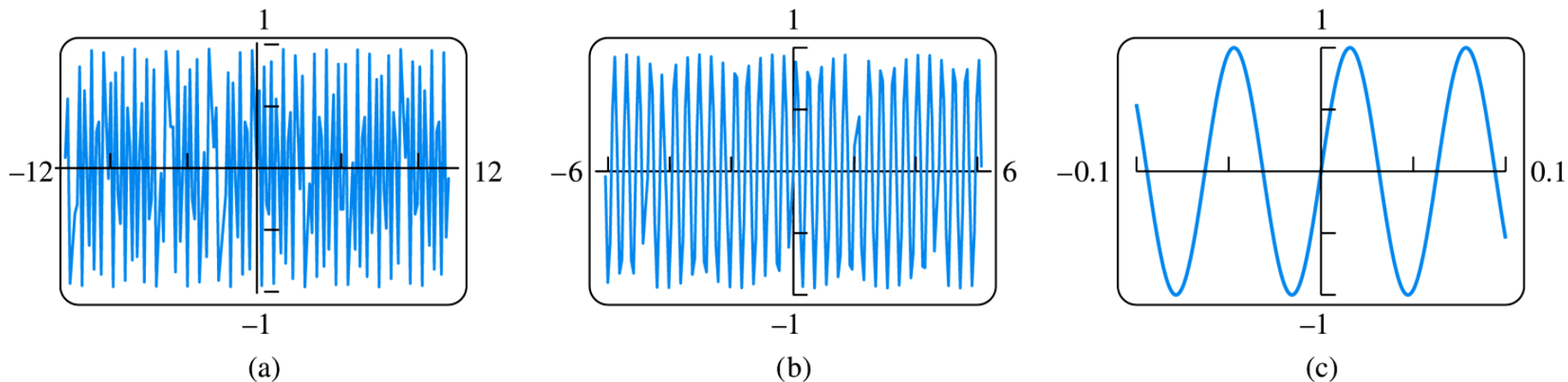


FIGURE 1.53 Graphs of the function $y = \sin 100x$ in three viewing windows. Because the period is $2\pi/100 \approx 0.063$, the smaller window in (c) best displays the true aspects of this rapidly oscillating function (Example 4).

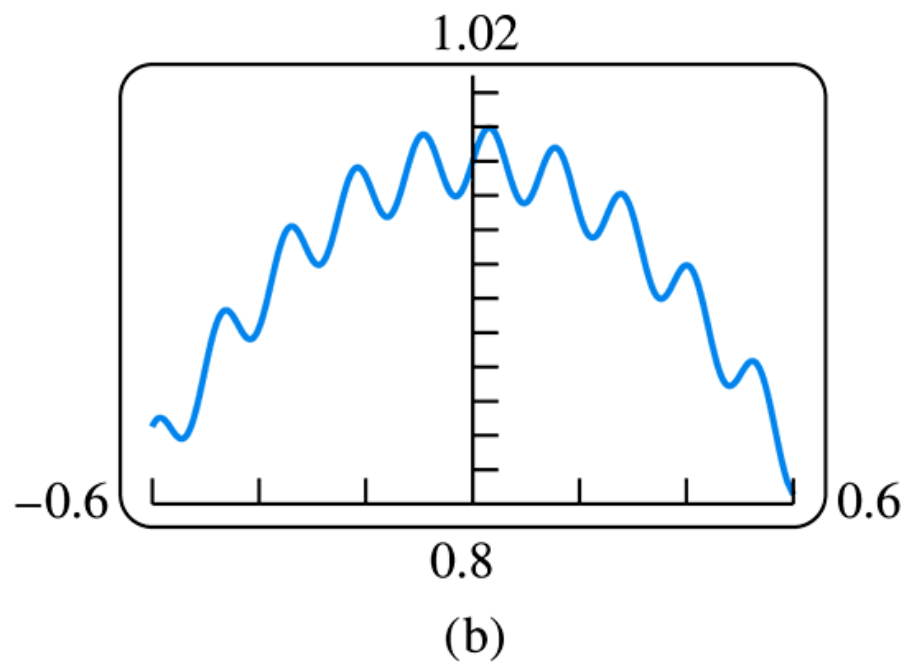
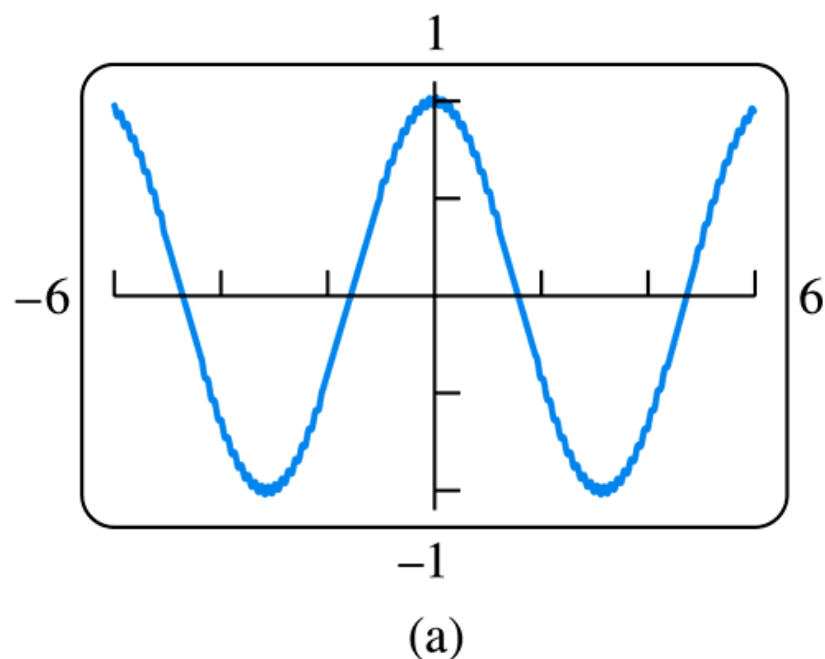


FIGURE 1.54 In (b) we see a close-up view of the function $y = \cos x + \frac{1}{50} \sin 50x$ graphed in (a). The term $\cos x$ clearly dominates the second term, $\frac{1}{50} \sin 50x$, which produces the rapid oscillations along the cosine curve. Both views are needed for a clear idea of the graph (Example 5).

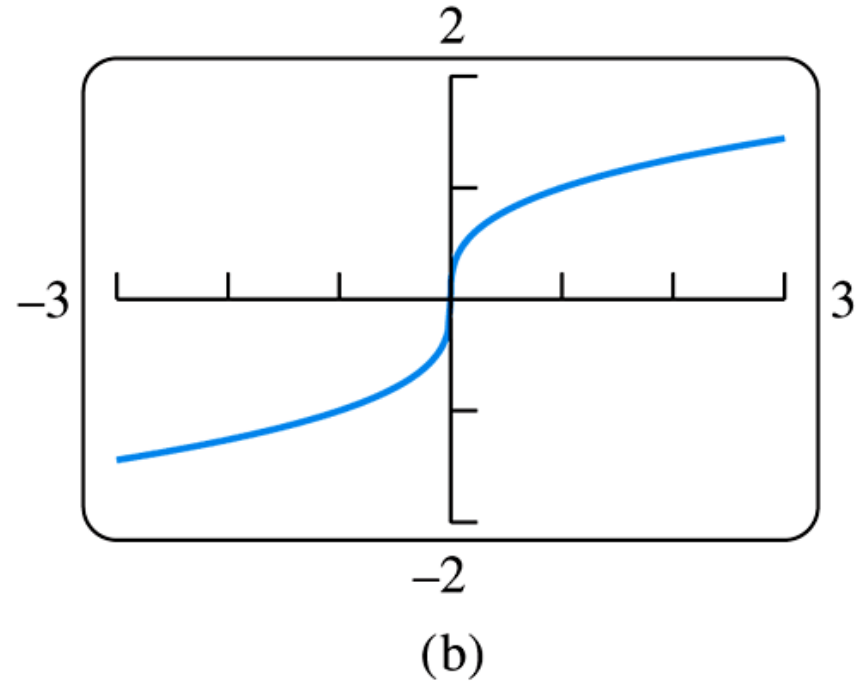
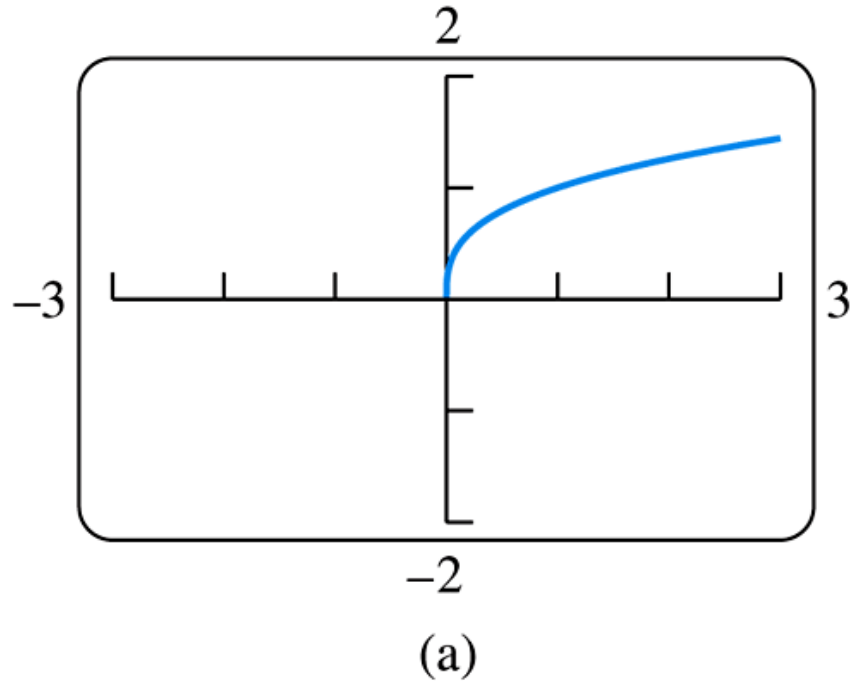


FIGURE 1.55 The graph of $y = x^{1/3}$ is missing the left branch in (a). In (b) we graph the function $f(x) = \frac{x}{|x|} \cdot |x|^{1/3}$, obtaining both branches. (See Example 6.)