Advanced Mathematical Analysis

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This lectures mainly based on course given by Pro. Vitaly Moroz in Swansea University.

This lecture notes are not intended to cover the entire content of the lectures

Reading books

1. Principles of Mathematical Analysis
   Walter Rudin 3^{rd} ed.

2. A First Course in Real Analysis
   M. H. Protter and C. B. Morry.

Contents

Chapter 4

Convergence in metric spaces

4.1 Definition of a convergent sequence

Definition 4.1.1. (Convergent sequence). Let (X, d) be a metric space. We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to the limit $x \in X$ if

$$\forall \epsilon > 0 \exists N(\epsilon) \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq N(\epsilon) \Longrightarrow d(x_n, x) < \epsilon. \tag{4.1}$$

In that case we write $x_n \longrightarrow x$ as $n \longrightarrow \infty$, or $\lim\limits_{n\longrightarrow\infty} x_n = x$

If a sequence has no limit we say that it diverges.

Remarks 4.1.2.

1. Comparing with the definition of a convergent sequence of real numbers from Real Analysis we see that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to the limit $x \in X$ if and only if the distance between x_n and x converges to zero in $\mathbb{R}$, i.e.

$$\lim_{n\longrightarrow\infty} d(x_n, x) = 0$$

in the sense of Real Analysis.

2. The limit of a convergent sequence is unique. Indeed, assume that $(x_n)_{n\in\mathbb{N}}$ converges simultaneously to $x \in X$ and to another limit $y \in$

X. Then using the triangle inequality we obtain

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since the right hand side converges to zero, we conclude that $d(x, y) = 0$ and hence $x = y$.

Definition 4.1.3. (Bounded set). Let (X, d) be a metric space and $E \subset X$ be a subset of X. We say that the set E is bounded in X if there exists a ball $B_R(x) \subset X$ such that $E \subset B_R(x)$. If the set E is not bounded then it is said to be unbounded. We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is bounded if $\{x_n\}_{n\in\mathbb{N}}$ is a bounded subset of X.

Example 4.1.4.

1. Let $X = \mathbb{R}$. The $[a, b]$ is bounded.
2. Let $X = \mathbb{R}^2$. The $(a, b) \times (a, b)$ is bounded.
3. Let $X = \mathbb{R}$. Then $\mathbb{N}$ is unbounded.
4. Let $X = \mathbb{R}^2$. Then $(a, b) \times \mathbb{R}$ is unbounded.

Theorem 4.1.5. *Let (X, d) be a metric space. If a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a limit in X then $(x_n)_{n\in\mathbb{N}}$ is bounded.*

Using the notion of convergence we could give an important characterisation of closed sets in terms of convergent sequences.

Theorem 4.1.6. *Let (X, d) be a metric space. A set $F \subset X$ is closed if and only if every convergent sequence $(x_n)_{n\in\mathbb{N}} \subset F$ has its limit in F, that is*

$$(x_n) \subset F \text{ and } \lim_{n\longrightarrow\infty} x_n = x \Longrightarrow x \in F \tag{4.2}$$

Definition 4.1.7. (Closure of a set). Let (X, d) be a metric space and $E \subset X$ a subset of X. The closure of the set E is the set

$$\mathrm{cl}(E) = \mathrm{int}(E) \cup \partial E$$

It is clear that the closure of a set E is always a closed set. The closure $\mathrm{cl}(E)$ can be characterised as the smallest closed set which contains E. In particular, if E is a closed set then $\mathrm{cl}(E) = E$.

Definition 4.1.8. (Dense set). Let (X, d) be a metric space and $E \subset X$ a nonempty subset of X. The set E is dense in X if $\mathrm{cl}(E) = X$.

Example 4.1.9. Let $X = \mathbb{R}$. Then $\mathrm{cl}((0,1)) = [0,1]$ and $\mathrm{cl}(\mathbb{Q}) = \mathbb{R}$. In particular, the set of all rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example 4.1.10. Let $B_R(a)$ be an open ball in the Euclidean space $\mathbb{R}^N$. Then

$$\mathrm{cl}(B_R(a)) = \bar{B}_R(a) = \{x \in X | d(x,a) \leq r\}$$

In particular, closed ball $\bar{B}_R(a)$ is a closed set in $\mathbb{R}^N$.

4.2 Convergence of sequences in $\mathbb{R}^N$

To avoid confusion between coordinates of the vector in $\mathbb{R}^N$ and elements of the sequence of vectors in $\mathbb{R}^N$, we sometime will be using the upper-script index notation $(x^{(n)})_{n\in\mathbb{N}}$ to denote a sequence of vectors in $\mathbb{R}^N$, where

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdots, x_N^{(n)})$$

For each $i = 1, \cdots, N$, the sequence $(x_i^{(n)})_{n\in\mathbb{N}} \subset \mathbb{R}$ is called the sequences of i-coordinates of the sequence $(x^{(n)})$.
The next result shows that a sequence of vectors in $\mathbb{R}^N$ converges if and only if each sequence of coordinates converges individually.

Proposition 4.2.1. *(Convergence in $\mathbb{R}^N$). Let $\mathbb{R}^N$ be the N-dimensional vector space with the standard Euclidean metric $d_2(x,y)$. A sequence*

$$(x^{(n)})_{n\in\mathbb{N}} = (x_1^{(n)}, x_2^{(n)}, \cdots, x_N^{(n)})_{n\in\mathbb{N}}$$

converges in $\mathbb{R}^N$ *to the limit*

$$x = (x_1, x_2, \cdots, x_N)$$

i.e. $\lim_{n\in\mathbb{N}} x^{(n)} = x$, *if and only if*
$\lim_{n\in\mathbb{N}} x_1^{(n)} = x_1,\ \lim_{n\in\mathbb{N}} x_2^{(n)} = x_2, \cdots,\ \lim_{n\in\mathbb{N}} x_N^{(n)} = x_N$

Remark 4.2.2. The same statement is true if instead of the Euclidean metric $d_2(x, y)$ we consider convergence in $\mathbb{R}^N$ with respect to the taxi-cab metric $d_1(x, y)$ or $\infty-$metric $d_\infty(x, y)$. In fact, one can show that the following three statements are equivalent:

(a) A sequence $(x^{(n)}) \subset \mathbb{R}^N$ converges to a vector $x \in \mathbb{R}^N$ in the metrics d_1

(b) A sequence $(x^{(n)}) \subset \mathbb{R}^N$ converges to a vector $x \in \mathbb{R}^N$ in the metrics d_2

(c) A sequence $(x^{(n)}) \subset \mathbb{R}^N$ converges to a vector $x \in \mathbb{R}^N$ in the metrics d_∞

Because of the equivalence of (a), (b) and (c) we say that the metrics d_1, d_2, d_∞ on $\mathbb{R}^N$ are equivalent, in the sense that they have the same classes of convergent sequences.

Example 4.2.3. Consider the following sequences in $\mathbb{R}^2$. Try to sketch on the plane $\mathbb{R}^2$ geometrical location of several points of the sequences in examples (1)-(5)

1. $(x^{(n)})_{n\in\mathbb{N}} = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$.

2. $(x^{(n)})_{n\in\mathbb{N}} = \left(\frac{1}{n}, \frac{1}{n^2}\right)$.

3. $(x^{(n)})_{n\in\mathbb{N}} = \left(\frac{1}{n}, \sqrt{n}\right)$.

4. $(x^{(n)})_{n\in\mathbb{N}} = \left(\cos(\frac{1}{n}), \sin(\frac{1}{n})\right)$.

5. $(x^{(n)})_{n\in\mathbb{N}} = (\sin(n\pi), \cos(n\pi))$.

4.3 Cauchy Sequences

Definition 4.3.1. (Cauchy sequence). Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called a Cauchy sequence if $\forall \epsilon > 0, \exists N(\epsilon) \in N$ such that

$$\forall n, m \in \mathbb{N}, n \geq N(\epsilon), m \geq N(\epsilon) \Longrightarrow d(x_m, x_n) < \epsilon$$

Theorem 4.3.2. *Let (X, d) be a metric space. Then every convergent sequence is also a Cauchy sequence.*

Definition 4.3.3. (Complete metric space). Let (X, d) be a metric space. We say that X is complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to the limit in X.

Example 4.3.4. The real line $\mathbb{R}$ with the standard metric $d_1(x, y) = |x-y|$ is complete.

Example 4.3.5. The set of all rational numbers $\mathbb{Q}$ with the standard metric $d_1(x, y) = |x - y|$ is not complete.

Example 4.3.6. The $N-$dimensional vector space $\mathbb{R}^N$ with any of the metrics d_1, d_2, d_∞ is complete. This follows from the completeness of the real line $\mathbb{R}$ via Proposition 4.2.1

4.4 Compact sets

Definition 4.4.1. (Compact set). Let (X, d) be a metric space and $K \subset X$ a subset of X. We say a set K is compact if every sequence $(x_n) \subset K$ contains at least one convergent subsequence (x_{n_k}) and

$$\lim_{k\longrightarrow\infty} x_{n_k} \longrightarrow x \in K.$$

Remark 4.4.2. In particular, we say that a metric space (X, d) is compact if every sequence $(x_n) \subset X$ contain at least one convergent subsequence.

Example 4.4.3. Every bounded closed interval $[a, b] \subset \mathbb{R}$ is compact. This is the Bolzano-Weierstrass Theorem. However, the real line $\mathbb{R}$ with the standard metric is not compact. For instance, the sequence $x_n = n$ does not contain any convergent subsequence.

Theorem 4.4.4. *Let (X, d) be a metric space. If a nonempty subset $K \subset X$ is compact then K is bounded and closed.*

Theorem 4.4.5. *(Heine-Borel). Let $\mathbb{R}^N$ be the Euclidean space. A subset $K \subset \mathbb{R}^N$ is compact if and only if K is bounded and closed.*

Corollary 4.4.6. *Let $\mathbb{R}^N$ be the Euclidean space. Then any bounded sequence $(x_n) \subset \mathbb{R}^N$ has a convergent subsequence.*

Remark 4.4.7. The same statement is true if instead of the Euclidean metric $d_2(x, y)$ we consider the taxi-cab metric $d_1(x, y)$ or $\infty-$metric $d_\infty(x, y)$. This is a consequence of the fact that all three metrics on $\mathbb{R}^N$ are equivalent, see Remark 4.2.2

Proposition 4.4.8. *Let (X, d) be a metric space and $K \subset X$ be a compact subset from X. If $M \subset K$ is closed then M is compact.6*

Proposition 4.4.9. *(from H.W-oct-1-sol. page 2)*
In $\mathbb{R}^N$ the intersection of arbitrary many compact set is compact

Proposition 4.4.10. *Let (X, d) be a metric space.*

1. *If $K_1, K_2, \cdots, K_n \subset X$ is a finite collection of compact sets, then the union $\bigcup_{i=1}^{n} K_i$ is also compact.*
2. *If $X \supset K_1 \supset K_2 \supset \cdots \supset K_n$ is a nested sequence of nonempty compact sets then the intersection $\bigcap_{i \in \mathbb{N}} K_i$ is nonempty.*