Advanced Mathematical Analysis

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This lectures mainly based on course given by Pro. Vitaly Moroz in Swansea University.

This lecture notes are not intended to cover the entire content of the lectures

Reading books

1. Principles of Mathematical Analysis
   Walter Rudin 3^{rd} ed.

2. A First Course in Real Analysis
   M. H. Protter   and   C. B. Morry.

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Chapter 2

Normed Spaces

2.1 Definition of a normed space

Recall that by a vector space over an algebraic field $\mathbb{F}$ we understand a nonempty set X equipped with two operations:

1. $(x, y) \longrightarrow x + y$ from $X \times X$ into X, (addition).
2. $(\lambda, x) \longrightarrow \lambda x$ from $\mathbb{F} \times X$ into X, (scalar multiplication).

Elements of the vector space X are called vectors, elements of the field $\mathbb{F}$ are called scalars (or sometimes numbers). Here we consider only real vector spaces with $\mathbb{F} = \mathbb{R}$. A norm on a vector space is function which roughly speaking has a meaning of the length of a vector. More precisely, we have the following definition.

Definition 2.1.1. (Normed space). Let X be a vector space. A norm on X is a function $\|\cdot\| : X \longrightarrow \mathbb{R}$, which satisfy the following properties:

(N1) $\|x\| = 0$ if and only if $x = 0$.

(N2) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$. (scalar multiplication)

(N3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. (Triangle inequality)

A normed space is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\|$ is a normed on X.

Example 2.1.2. (Positivity of the norm). Let $(X, \|\cdot\|)$ be a normed space. Prove that

$$(N1') \qquad \|x\| \geq 0 \qquad \forall x \in X. \tag{2.1}$$

Example

$(\mathbb{R}, |\cdot|)$ is normed space.

Theorem 2.1.3. *Let $(X, \|\cdot\|)$ be a normed space. Then*

$$d(x, y) := \|x - y\| \tag{2.2}$$

defines a metric on X, i.e. (X, d) is a metric space.

Remark 2.1.4. Theorem 2.1.3 states that every normed space is also a metric space with the induced metric. However, not every metric space can be made into a normed space. For example, the metric space of all positive rational numbers Q_+ with the metric $d(x, y) = |log(\frac{x}{y})|$ from Example 1.1.4 is not a normed space, simply because Q_+ is not a vector space.

2.2 Vector space $\mathbb{R}^N$

The space $\mathbb{R}^N$ of N-vectors of real numbers, defined in Example 1.1.6 is a vector space. The number N is called the dimension of the space $\mathbb{R}^N$. For vectors $x, y \in \mathbb{R}^N$ and scalar $\lambda \in \mathbb{R}$, the addition and scalar multiplication are defined as follows:

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_N + y_N)$$

$$\lambda x = (\lambda x_1, \lambda x_2, \cdots, \lambda x_N)$$

Proposition 2.2.1. *(The Cauchy-Schwarz inequality). For any $x, y \in \mathbb{R}^N$,*

$$|x \cdot y| \leq \|x\|\|y\|. \tag{2.3}$$

Example 2.2.2. (Euclidean norm on $\mathbb{R}^N$). The function

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_N|^2} = \sqrt{\sum_{i=1}^{N} |x_i|^2}$$

defines the Euclidean norm on $\mathbb{R}^N$.

Corollary 2.2.3. $(\mathbb{R}^N, d_2)$ *is metric space*

Example 2.2.4. (Taxi-cab and ∞ norm on R^N). Let $\mathbb{R}^N$ be the N-dimensional vector space as before. We define the taxi-cab norm $\|\cdot\|_1 : \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$\|x\|_1 = |x_1| + \cdots + |x_N|$$

and infinity norm $\|\cdot\|_\infty : \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$\|x\|_\infty = \max\{|x_1|, \cdots, |x_N|\}$$

Example

Explain why $\|x\| = \min\{x_1, x_2\}$ does not define a norm on $\mathbb{R}^2$.

2.3 Vector space of continuous functions $C([a, b])$

Let $C([a, b])$ be the vector space of all continuous functions on the closed interval $[a, b]$. For functions $f, g \in C([a, b])$ and scalar $\lambda \in \mathbb{R}$, the addition and scalar multiplication are defined in a natural pointwise way:

$$(f + g)(x) = f(x) + g(x)$$

$$(\lambda f)(x) = \lambda f(x)$$

Example 2.3.1. (Uniform convergence norm on $C([a, b])$). The function

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$

is a norm on $C([a, b])$, known as a uniform convergence norm, or ∞-norm on $C([a, b])$.