

# Advanced Mathematical Analysis

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This lectures mainly based on course given by Pro. Vitaly Moroz in  
Swansea University.

This lecture notes are not intended to cover the entire content of the  
lectures

# Reading books

1. Principles of Mathematical Analysis  
Walter Rudin     3<sup>rd</sup> ed.
2. A First Course in Real Analysis  
M. H. Protter   and   C. B. Morry.

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# Chapter 1

## Metric space

### 1.1 Definition and Examples

**Definition 1.1.1.** (Metric space). Let  $X$  be a nonempty set. A metric (or a distance)  $d$  on  $X$  is a function

$$d: X \times X \longrightarrow \mathbb{R}$$

which satisfies the following properties:

(M1)  $d(x, y) = 0$  if and only if  $x = y$ ,

(M2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ , (Symmetry)

(M3)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ , (Triangle inequality)

The pair  $(X, d)$  is called a metric space.

**Example 1.1.2.** (Positivity of the metric). Prove that any metric  $d: X \times X \longrightarrow \mathbb{R}$  satisfy the following property:

$$(M1') \quad d(x, y) \geq 0 \quad \forall x, y \in X. \quad (\text{Positivity})$$

**Example 1.1.3.** (The Real line). Let  $\mathbb{R}$  be the set of all real numbers.

Define a metric  $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$  by

$$d(x, y) = |x - y| \tag{1.1}$$

Then  $(\mathbb{R}, d)$  is a metric space. We refer to this metric as the standard metric on  $\mathbb{R}$ .

**Example 1.1.4.** Prove that the set of all positive rational numbers  $\mathbb{Q}_+$  with the metric  $d(x, y) = |\log(\frac{x}{y})|$  is a metric space.

**Example**

$d(x, y) = |\log(\frac{x}{y})|$  is not metric on  $\mathbb{Q}$ .

**Example**

$d(x, y) = |\log(\frac{|x|}{|y|})|$  is not metric on  $\mathbb{Q}$ .

**Example 1.1.5.** (Discreet metric). Let  $X$  be an arbitrary set. Define discreet metric  $d : X \times X \longrightarrow \mathbb{R}$  by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Prove that  $(X, d)$  is a metric space.

**Example 1.1.6.** (Euclidean space  $\mathbb{R}^N$ ). Let  $N \in \mathbb{N}$  be a natural number and let  $\mathbb{R}^N$  be the space of N-vectors of real numbers:

$$\mathbb{R}^N = \{f(x_1, x_2, \dots, x_N) | x_1, \dots, x_N \in \mathbb{R}\}$$

When we write  $x \in \mathbb{R}^N$  this means  $x$  is an N-vector, that is  $x = (x_1, x_2, \dots, x_N)$ .

We define the Euclidean metric  $d_2 : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$  by

$$d_2(x, y) = \sqrt{|x_1 - y_1|^2 + \dots + |x_N - y_N|^2} \tag{1.2}$$

Then  $(\mathbb{R}^N, d_2)$  is a metric space, which we call Euclidean space of dimen-

sion  $N$ .

**Example 1.1.7.** (Taxi-cab metric on  $\mathbb{R}^N$ ). Let  $N \in \mathbb{N}$  be a natural number and let  $\mathbb{R}^N$  be the space of  $N$ -vectors as before. We define the taxi-cab metric  $d_1 : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$  by

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_N - y_N| \quad (1.3)$$

Then  $(\mathbb{R}^N, d_1)$  is a metric space.

**Example 1.1.8.** ( $\infty$ -metric on  $\mathbb{R}^N$ ). Again let  $N \in \mathbb{N}$  be a natural number and let  $\mathbb{R}^N$  be as before. We define the sup-norm metric  $d_\infty : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$  by

$$d_\infty(x, y) = \max\{|x_1 - y_1|, \cdots, |x_N - y_N|\} \quad (1.4)$$

Then  $(\mathbb{R}^N, d_\infty)$  is a metric space.

*Remark 1.1.9.* The  $d_1, d_2$  and  $d_\infty$  metrics on  $\mathbb{R}^N$  are special cases of the more general  $d_p$ -metric on  $\mathbb{R}^N$ ,

$$d_p(x, y) = (|x_1 - y_1|^p + \cdots + |x_N - y_N|^p)^{\frac{1}{p}} \quad (1.5)$$

where  $p \in [1, \infty)$ . Note that  $d_\infty < d_2 < d_1$ .

**Example 1.1.10.** (Metric of uniform convergence on  $C([a, b])$ ). Let  $C([a, b])$  denote the set of continuous functions  $f : [a, b] \longrightarrow \mathbb{R}$ ,

$$C([a, b]) = \{f : [a, b] \longrightarrow \mathbb{R} \mid f \text{ is continuous on } [a, b]\}. \quad (1.6)$$

Then

$$d_\infty(f, g) = \max_{x \in [a, b]} |f(x) - g(x)| \quad (1.7)$$

is a metric on  $C([a, b])$ . This metric is known as metric of uniform convergence, or  $\infty$ -metric on  $C([a, b])$ .

**Example 1.1.11.** Let  $f(x) = x^2$  and  $g(x) = x^3$ . Find the distances  $d_\infty(f, g)$  in  $C([0, 1])$ , and in  $C([-1, 1])$ .