Advanced Mathematical Analysis

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This lectures mainly based on course given by Pro. Vitaly Moroz in Swansea University.

This lecture notes are not intended to cover the entire content of the lectures

Reading books

1. Principles of Mathematical Analysis
   Walter Rudin 3^{rd} ed.

2. A First Course in Real Analysis
   M. H. Protter and C. B. Morry.

Contents

Chapter 1

Metric space

1.1 Definition and Examples

Definition 1.1.1. (Metric space). Let X be a nonempty set. A metric (or a distance) d on X is a function

$$d\colon X \times X \longrightarrow R$$

which satisfies the following properties:

(M1) $d(x, y) = 0$ if and only if $x = y$,

(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$, (Symmetry)

(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$, (Triangle inequality)

The pair (Xd) is called a metric space.

Example 1.1.2. (Positivity of the metric). Prove that any metric $d\colon X \times X \longrightarrow \mathbb{R}$ satisfy the following property:

$$(M1') \qquad d(x, y) \geq 0 \quad \forall x, y \in X. \qquad \text{(Positivity)}$$

Example 1.1.3. (The Real line). Let $\mathbb{R}$ be the set of all real numbers.

Define a metric $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$d(x, y) = |x - y| \tag{1.1}$$

Then $(\mathbb{R}, d)$ is a metric space. We refer to this metric as the standard metric on $\mathbb{R}$.

Example 1.1.4. Prove that the set of all positive rational numbers $\mathbb{Q}_+$ with the metric $d(x, y) = |log(\frac{x}{y})|$ is a metric space.

Example

$d(x, y) = |log(\frac{x}{y})|$ is not metric on $\mathbb{Q}$.

Example

$d(x, y) = |log(\frac{|x|}{|y|})|$ is not metric on $\mathbb{Q}$.

Example 1.1.5. (Discreet metric). Let X be an arbitrary set. Define discreet metric $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Prove that (X, d) is a metric space.

Example 1.1.6. (Euclidean space $\mathbb{R}^N$). Let $N \in \mathbb{N}$ be a natural number and let $\mathbb{R}^N$ be the space of N-vectors of real numbers:

$$\mathbb{R}^N = \{f(x_1, x_2, \cdots, x_N)|x_1, \cdots, x_N \in \mathbb{R}\}$$

When we write $x \in \mathbb{R}^N$ this means x is an N-vector, that is $x = (x_1, x_2, \cdots, x_N)$. We define the Euclidean metric $d_2 : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$d_2(x, y) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_N - y_N|^2} \tag{1.2}$$

Then $(\mathbb{R}^N, d_2)$ is a metric space, which we call Eucleadean space of dimension N.

Example 1.1.7. (Taxi-cab metric on $\mathbb{R}^N$). Let $N \in \mathbb{N}$ be a natural number and let $\mathbb{R}^N$ be the space of N-vectors as before. We define the taxi-cab metric $d_1 : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_N - y_N| \tag{1.3}$$

Then $(\mathbb{R}^N, d_1)$ is a metric space.

Example 1.1.8. (∞-metric on $\mathbb{R}^N$). Again let $N \in \mathbb{N}$ be a natural number and let $\mathbb{R}^N$ be as before. We define the sup-norm metric $d_\infty : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$d_\infty(x, y) = \max\{|x_1 - y_1|, \cdots, |x_N - y_N|\} \tag{1.4}$$

Then $(\mathbb{R}^N, d_\infty)$ is a metric space.

Remark 1.1.9. The d_1, d_2 and d_∞ metrics on $\mathbb{R}^N$ are special cases of the more general d_p-metric on $\mathbb{R}^N$,

$$d_p(x, y) = (|x_1 - y_1|^p + \cdots + |x_N - y_N|^p)^{\frac{1}{p}} \tag{1.5}$$

where $p \in [1, \infty)$. Note that $d_\infty < d_2 < d_1$.

Example 1.1.10. (Metric of uniform convergence on $C([a, b])$). Let $C([a, b])$ denote the set of continuous functions $f : [a, b] \longrightarrow \mathbb{R}$,

$$C([a, b]) = \{f : [a, b] \longrightarrow \mathbb{R} | f \text{ is continuous on } [a, b]\}. \tag{1.6}$$

Then

$$d_\infty(f, g) = \max_{x \in [a,b]} |f(x) - g(x)| \tag{1.7}$$

is a metric on $C([a,b])$. This metric is known as metric of uniform convergence, or ∞-metric on $C([a,b])$.

Example 1.1.11. Let $f(x) = x^2$ and $g(x) = x^3$. Find the distances $d_\infty(f,g)$ in $C([0,1])$, and in $C([-1,1])$.