Chapter 4

General Motion of a particle in Three Dimension

In **3D**, the vector form of the equation of motion is ;

$$\mathbf{F} = m\ddot{\mathbf{r}}$$

This vector equation is equivalent to three scalar equations in Cartesian coordinates

$$F_x = m\ddot{x} \qquad F_y = m\ddot{y} \qquad F_z = m\ddot{z}$$

In order to develop a powerful analytical technique that can be applied in such case, we need to use the **Work** principle.

In **3D**;

$$W = \int \mathbf{F} \cdot d\mathbf{r} = \int dT = \Delta T$$

And, if the **force** acting on a particle was *conservative*, then

$$W = \int \mathbf{F}.\, d\mathbf{r} = -\int dV(\mathbf{r}) = -\Delta V$$

A Force is "Conservative" if:

Conservative Force ≡ The work done by that force depends only on initial & final conditions & not on path taken between the initial & final positions of the object.

Non-Conservative Force ≡ The work done by that force depends on the path taken between the initial & final positions of the mass.
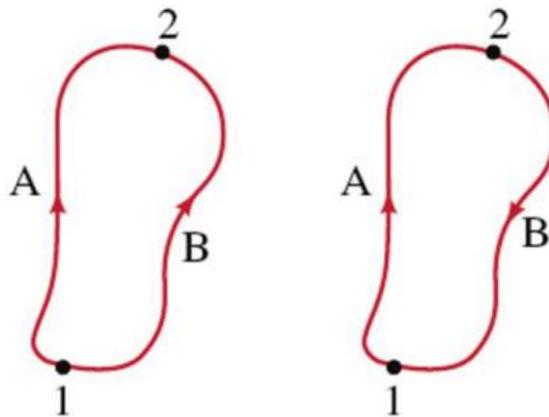
Hence, *the conservation of total energy principle is applicable.*

$$E_{tot} = V(A) + T(A) = V(B) + T(B) = constant$$

Therefore, when **F** is *conservative* the work, *W*, is zero for any simple closed path:

$$\oint F.dr = 0$$

If a force is **Conservative**, a **Potential Energy** *CAN* be defined.

But, if a force is **Non-Conservative**, a **Potential Energy** *CAN NOT* be defined!!

Law of Conservation of Total Energy

In any process, total energy is neither created nor destroyed.

Energy can be transformed from one form to another & from one object to another, but the Total Amount Remains Constant.

The most familiar conservative forces are *gravity, the electric force, and spring force.*

The Conditions that a Force to be Conservative

❏ The two conditions for a force to be conservative are

> **Conditions for a Force to be Conservative**
> A force **F** acting on a particle is conservative if and only if it satisfies two conditions:
>
> 1. F depends only on the <u>particle's position **r**</u> (and not on the velocity **v**, or the time *t*, or any other variable); that is, **F = F(r)**.
> 2. The work done by that force depends only on initial & final conditions & not on path taken between the initial & final positions of the object.

❏ It turns out that there is an easy way to check whether a force has the second property, using a concept from vector calculus. It can be shown that the work it does is independent of the path, if and only if $\nabla \times F = 0$ everywhere. The quantity $\nabla \times F$ is called the **curl** of **F**, or just "curl **F**," or "del cross **F**."

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

❏ It follows the usual rules for the cross product.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

Definition (2):

Therefore, when **F** is *conservative* the work, *W*, is zero for any simple closed path:
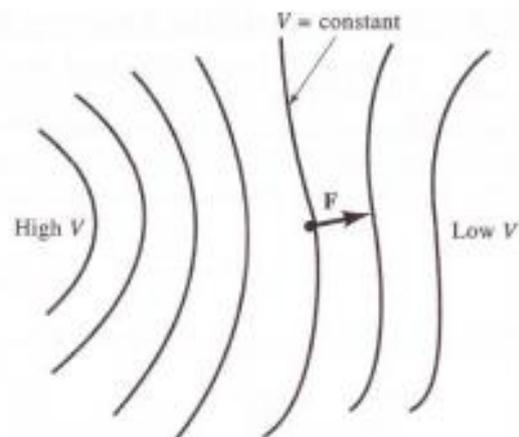
$$\oint F.dr = 0 \qquad \textbf{(1)}$$

In **3D**, we can express a conservative force as

$$\mathbf{F} = -\mathbf{i}\frac{\partial V}{\partial x} - \mathbf{j}\frac{\partial V}{\partial y} - \mathbf{k}\frac{\partial V}{\partial z}$$

This equation can be written as;

$$\mathbf{F} = -\nabla V \qquad \textbf{(2)}$$

The meaning of the **negative sign** is that the particle is forced to move in the direction of **decreasing potential energy**

Independent (Separable) Forces

Suppose that we have a force where the coordinates x, y, and z are independent variables

$$\mathbf{F} = \mathbf{i}Fx(x) + \mathbf{j}Fy\,(y) + \mathbf{k}Fz(z)$$

Forces of this type are *separable*. The curl of such a force is identically zero:

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x(x) & F_y(y) & F_z(z) \end{vmatrix} = 0$$

In this case the field is **conservative** and the **equations of motion** for each component can be solved by the methods described under *rectilinear motion* in Chapter 2.

EXAMPLE 4.2.1

Given the two-dimensional potential energy function

$$V(\mathbf{r}) = V_0 - \tfrac{1}{2}k\delta^2 e^{-r^2/\delta^2}$$

where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y$ and V_0, k, and δ are constants, find the force function.

Solution:

We first write the potential energy function as a function of x and y,

$$V(x,y) = V_0 - \tfrac{1}{2}k\delta^2 e^{-(x^2+y^2)/\delta^2}$$

and then apply the gradient operator:

$$\mathbf{F} = -\nabla V = -\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right)V(x,y)$$

$$= -k(\mathbf{i}x + \mathbf{j}y)e^{-(x^2+y^2)/\delta^2}$$
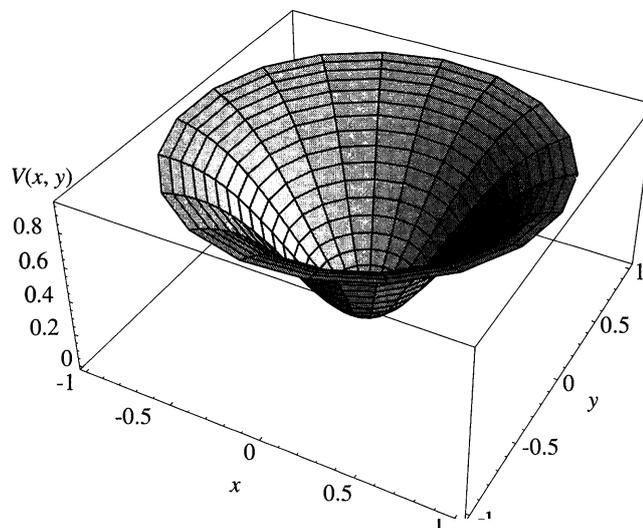
$$= -k\mathbf{r}e^{-r^2/\delta^2}$$

Figure 4.2.2a The potential energy function $V(x, y) = V_0 - \tfrac{1}{2}k\delta^2 e^{-(x^2+y^2)/\delta^2}$.

EXAMPLE 4.2.2

Suppose a particle of mass m is moving in the above force field, and at time $t = 0$ the particle passes through the origin with speed v_0. What will the speed of the particle be at some small distance away from the origin given by $\mathbf{r} = \mathbf{e}_r\Delta$, where $\Delta \ll \delta$?

Solution:

The force is conservative, because a potential energy function exists. Thus, the total energy $E = T + V = $ constant,

$$E = \tfrac{1}{2}mv^2 + V(\mathbf{r}) = \tfrac{1}{2}mv_0^2 + V(0)$$

and solving for v, we obtain

$$v^2 = v_0^2 + \frac{2}{m}[V(0) - V(\mathbf{r})]$$

$$= v_0^2 + \frac{2}{m}\left[\left(V_0 - \tfrac{1}{2}k\delta^2\right) - \left(V_0 - \tfrac{1}{2}k\delta^2 e^{-\Delta^2/\delta^2}\right)\right]$$

$$= v_0^2 - \frac{k\delta^2}{m}[1 - e^{-\Delta^2/\delta^2}]$$

$$\approx v_0^2 - \frac{k\delta^2}{m}[1 - (1 - \Delta^2/\delta^2)]$$

$$= v_0^2 - \frac{k}{m}\Delta^2$$

The potential energy is a quadratic function of the displacement Δ from the origin for small displacements, so this solution reduces to the conservation of energy for the simple harmonic oscillator

EXAMPLE 4.2.3

Is the force field $\mathbf{F} = \mathbf{i}xy + \mathbf{j}xz + \mathbf{k}yz$ conservative? The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xz & yz \end{vmatrix} = \mathbf{i}(z - x) + \mathbf{j}0 + \mathbf{k}(z - x)$$

The final expression is not zero for all values of the coordinates; hence, the field is *not* conservative.

EXAMPLE 4.2.4

For what values of the constants a, b, and c is the force $\mathbf{F} = \mathbf{i}(ax + by^2) + \mathbf{j}cxy$ conservative? Taking the curl, we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax + by^2 & cxy & 0 \end{vmatrix} = \mathbf{k}(c - 2b)y$$

This shows that the force is conservative, provided $c = 2b$. The value of a is immaterial.

EXAMPLE 4.2.5

Show that the inverse-square law of force in three dimensions $\mathbf{F} = (-k/r^2)\mathbf{e}_r$ is conservative by the use of the curl. Use spherical coordinates. The curl is given in Appendix F as

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta r & \mathbf{e}_\phi r \sin\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & rF_\phi \sin\theta \end{vmatrix}$$

We have $F_r = -k/r^2$, $F_\theta = 0$, $F_\phi = 0$. The curl then reduces to

$$\nabla \times \mathbf{F} = \frac{\mathbf{e}_\theta}{r \sin\theta} \frac{\partial}{\partial \phi}\left(\frac{-k}{r^2}\right) - \frac{\mathbf{e}_\phi}{r} \frac{\partial}{\partial \theta}\left(\frac{-k}{r^2}\right) = 0$$

which, of course, vanishes because both partial derivatives are zero. Thus, the force in question is conservative.