3.6 Forced Harmonic Motion: Resonance

Let us exert an external periodic force upon a damped harmonic oscillator. When the driving frequency is close to the natural frequency ω_0 of the oscillator, a remarkable phenomenon, called **resonance**, occurs.

Suppose that the applied force has the form of $F_0 e^{i\omega t}$. Thus, the equation of motion is

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t}$$

solution of the complex exponential type;

$$x(t) = Ae^{i(\omega t - \phi)}$$

where the amplitude A and phase difference ϕ are constants to be determined. If this "guess" is correct, we must have

$$m\frac{d^2}{dt^2}Ae^{i(\omega t-\phi)} + c\frac{d}{dt}Ae^{i(\omega t-\phi)} + kAe^{i(\omega t-\phi)} = F_0 e^{i\omega t}$$

and canceling the common factor $e^{i\omega t}$, we find

$$-m\omega^2 A + i\omega c A + kA = F_0 e^{i\phi} = F_0(\cos\phi + i\sin\phi)$$

Equating the real and imaginary parts yields the two equations

$$A(k - m\omega^2) = F_0\cos\phi$$
$$c\omega A = F_0\sin\phi$$

Upon dividing the second by the first and using the identity $\tan\phi = \sin\phi/\cos\phi$, we obtain the following relation for the phase angle:

$$\tan\phi = \frac{c\omega}{k - m\omega^2}$$

By squaring both sides of Equations

$$A^2(k - m\omega^2)^2 = F_o^2 cos^2\phi$$
$$A^2\omega^2 c^2 = F_o^2 sin^2\phi$$

and adding,

$$A^2(k - m\omega^2)^2 + A^2\omega^2 c^2 = F_o^2 cos^2\phi + F_o^2 sin^2\phi$$

Using the identity, $\sin^2\phi + \cos^2\phi = 1$, we find

$$A^2(k - m\omega^2)^2 + c^2\omega^2 A^2 = F_0^2$$

We can then solve for A, the amplitude of the steady-state oscillation, as a function of the driving frequency:

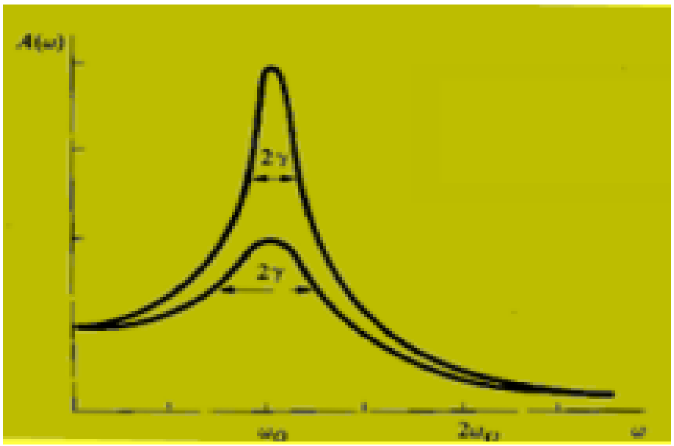
$$A(\omega) = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

In another form by using $\omega_0^2 = k/m$ and $\gamma = c/2m$,

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

$$A(\omega) = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\right]^{1/2}}$$

A plot of $A(\omega)$ versus ω shows that the amplitude presumes a maximum value at a certain applied frequency ω_r called the amplitude resonant frequency, or ***resonant frequency***.



To find ω_r we calculate $\frac{dA(\omega)}{d\omega} = 0$. Thus;

$$\frac{dA(\omega)}{d\omega} = \frac{2F_o\omega}{m}\frac{(\omega_o^2 - \omega^2 - 2\gamma^2)}{((\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2)^{3/2}} = 0$$

we obtain

$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$

ω_r approaches ω_0 as γ, the damping term, goes to zero. Because the angular frequency of the freely running damped oscillator is given by $\omega_d = (\omega_0^2 - \gamma^2)^{1/2}$, we have

$$\omega_r^2 = \omega_d^2 - \gamma^2$$

Amplitude of Oscillation at the Resonance Peak

The steady-state amplitude at the resonant frequency, which we call A_{max}, is obtained from

$$A_{max} = A(\omega_r) = \frac{F_o/m}{2\gamma\sqrt{\omega_o^2 - \gamma^2}}$$

In the case of weak damping, we can neglect γ^2 and write

$$A_{max} \simeq \frac{F_0}{2\gamma m \omega_0}$$

When the damping is weak:

From $\omega_r^2 = \omega_0^2 - 2\gamma^2$ $\quad\quad \omega_r^2 = \omega_d^2 - \gamma^2$

If $\gamma << \omega_o$, the resonant frequency ω_r, the freely running damped oscillator frequency ω_d, and the natural frequency ω_o of the oscillator are essentially identical.

I.e; $\omega_r \approx \omega_d \approx \omega_o$

Problems

3.1 A guitar string vibrates harmonically with a frequency of 512 Hz (one octave above middle C on the musical scale). If the amplitude of oscillation of the centerpoint of the string is 0.002 m (2 mm), what are the maximum speed and the maximum acceleration at that point?

$$x = 0.002\sin\left[2\pi\left(512 s^{-1}\right)t\right][m]$$

$$\dot{x}_{\max} = (0.002)(2\pi)(512)\left[\frac{m}{s}\right] = 6.43\left[\frac{m}{s}\right]$$

$$\ddot{x}_{\max} = (0.002)(2\pi)^2(512)^2\left[\frac{m}{s^2}\right] = 2.07\times10^4\left[\frac{m}{s^2}\right]$$

3.9 Show that the ratio of two successive maxima in the displacement of a damped harmonic oscillator is constant.

$$x = e^{-\gamma t}A\cos(\omega_d t - \phi)$$

$$\frac{dx}{dt} = -e^{-\gamma t}A\omega_d\sin(\omega_d t - \phi) - \gamma e^{-\gamma t}A\cos(\omega_d t - \phi)$$

maxima at $\frac{dx}{dt} = 0 = \omega_d\sin(\omega_d t - \phi) + \gamma\cos(\omega_d t - \phi)$

$$\tan(\omega_d t - \phi) = -\frac{\gamma}{\omega_d}$$

thus the condition of relative maximum occurs every time that t increases by $\frac{2\pi}{\omega_d}$:

$$t_{i+1} = t_i + \frac{2\pi}{\omega_d}$$

For the i th maximum: $x_i = e^{-\gamma t_i}A\cos(\omega_d t_i - \phi)$

$$x_{i+1} = e^{-\gamma t_{i+1}}A\cos(\omega_d t_{i+1} - \phi) = e^{-\gamma\frac{2\pi}{\omega_d}}x_i$$

$$\frac{x_i}{x_{i+1}} = e^{\gamma T_d}$$

3.10 A damped harmonic oscillator with $m = 10$ kg, $k = 250$ N/m, and $c = 60$ kg/s is subject to a driving force given by $F_0 \cos \omega t$, where $F_0 = 48$ N.

(a) What value of ω results in steady-state oscillations with maximum amplitude? Under this condition:

(b) What is the maximum amplitude?

(c) What is the phase shift?

(a) $\gamma = \frac{c}{2m} = 3\,s^{-1}$ $\qquad \omega_\circ^2 = \frac{k}{m} = 25\,s^{-2}$

$\omega_d^2 = \omega_\circ^2 - \gamma^2 = 16\,s^{-2}$ $\qquad \omega_r^2 = \omega_d^2 - \gamma^2 = 7\,s^{-2}$

$\therefore \omega_r = \sqrt{7}\,s^{-1}$

(b) $A_{max} = \frac{F_\circ}{C\omega_d} = \frac{48}{60.4}\,m = 0.2\,m$

(c) $\tan\phi = \frac{2\gamma\omega_r}{(\omega_\circ^2 - \omega_r^2)} = \frac{2\gamma\omega_r}{2\gamma^2} = \frac{\omega_r}{\gamma} = \frac{\sqrt{7}}{3}$ $\qquad \therefore \phi \approx 41.4^\circ$

3.12 The frequency f_d of a damped harmonic oscillator is 100 Hz, and the ratio of the amplitude of two successive maxima is one half.

(a) What is the undamped frequency f_0 of this oscillator?

(b) What is the resonant frequency f_r?

$$e^{-\gamma T_d} = \frac{1}{2}$$

$$\gamma = \frac{1}{T_d}\ln 2 = f_d \ln 2$$

(a) $\omega_d = (\omega_\circ^2 - \gamma^2)^{\frac{1}{2}}$

So, $\omega_\circ = (\omega_d^2 + \gamma^2)^{\frac{1}{2}}$

$$f_\circ = \left[f_d^2 + \left(\frac{\gamma}{2\pi}\right)^2\right]^{\frac{1}{2}} = f_d\left[1 + \left(\frac{\ln 2}{2\pi}\right)^2\right]^{\frac{1}{2}}$$

$$f_\circ = 100.6\,Hz$$

(b) $\omega_r = \left(\omega_d^2 - \gamma^2\right)^{\frac{1}{2}}$

$$f_r = \left[f_d^2 - \left(\frac{\gamma}{2\pi}\right)^2\right]^{\frac{1}{2}} = f_d\left[1 - \left(\frac{\ln 2}{2\pi}\right)^2\right]^{\frac{1}{2}}$$

$$f_r = 99.4\,Hz$$