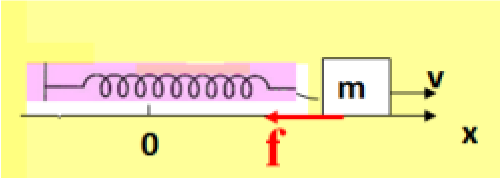
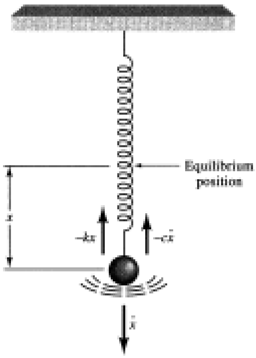
Damped harmonic motion





Up to this point we have assumed that no frictional force act on the system.

For real oscillator, there may be friction, air resistance act on the system, the amplitude will decrease.

This loss in amplitude is called "*damping*" and the motion is called "damped harmonic motion".

Friction or other sources of external work can lead to a **loss of energy,** (known as dissipation), from an oscillating system. This phenomenon is referred to as **damping**.

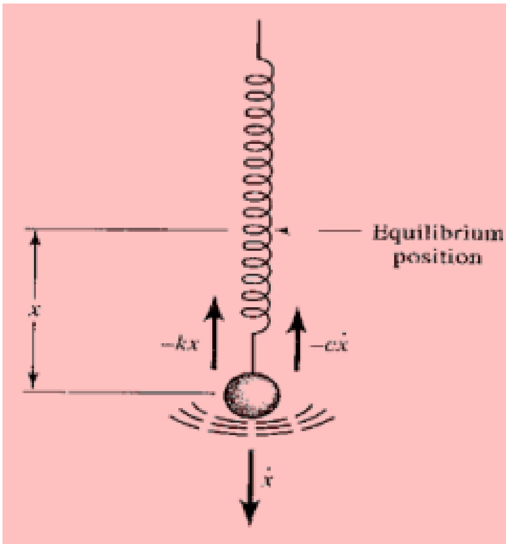
Damping has two principal effects on the oscillating system. It

- **decreases the amplitude of the oscillations and**
- **decreases the frequency (increases the period) of oscillations.**

Damped Harmonic Motion

Consider an object of mass m is supported by a light spring of constant k. We assume that there is a viscous retarding force (-cv) that is a linear function of the velocity.

The differential equation of motion is, therefore,



$$m\ddot{x} + c\dot{x} + kx = 0$$

Or,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

If we introduce the damping factor (γ) defined as

$$\gamma = \frac{c}{2m}$$

Then

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

Simple sine or cosine solutions do not work, because of the presence of the velocity-dependent term.

The suitable solution for this case is;

$$x(t) = A_1 e^{-(\gamma-q)t} + A_2 e^{-(\gamma+q)t}$$

where

$$q = \sqrt{\gamma^2 - \omega_0^2}$$

Let D be the differential operator d/dt.

$$\left[D^2 + 2\gamma D + \omega_0^2\right]x = 0$$

$$\left[D+\gamma-\sqrt{\gamma^2-\omega_0^2}\right]\left[D+\gamma+\sqrt{\gamma^2-\omega_0^2}\right]x = 0$$

There are three possible situations:

I. q real > 0 (**Overdamping**)
II. q real = 0 (**Critical damping**)
III. q imaginary (**Underdamping**)

I. Overdamped case:

Both exponents are real. The constants A1 and A2 are determined by the initial conditions. The motion is an exponential decay with two different decay constants, $(\gamma - q)$ and $(\gamma + q)$. **The mass will be prevented from oscillating by the strong damping force.**

II. Critical damping case:

Here $q = 0$. The two exponents are each equal to γ.

$$(D+\gamma)(D+\gamma)x = 0$$

we make the substitution $u = (D+\gamma)x$, which gives

$$(D+\gamma)u = 0$$

$$u = Ae^{-\gamma t}$$

Equating this to $(D+\gamma)x$, the final solution is obtained as follows:

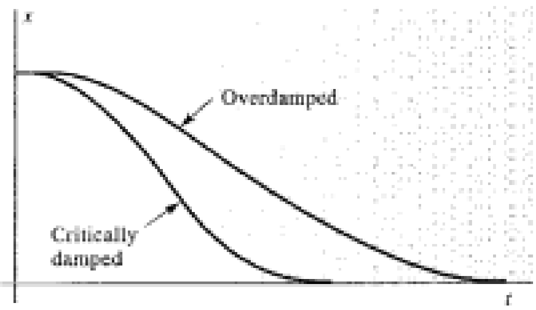
$$Ae^{-\gamma t} = (D+\gamma)x$$

$$A = e^{\gamma t}(D+\gamma)x = D(xe^{\gamma t})$$

$$\therefore\ xe^{\gamma t} = At + B$$

$$x(t) = Ate^{-\gamma t} + Be^{-\gamma t}$$

As in case I, **the motion is a returning to equilibrium with no oscillation.**



EXAMPLE 3.4.1

An automobile suspension system is critically damped, and its period of free oscillation with no damping is 1 s. If the system is initially displaced by an amount x_0 and released with zero initial velocity, find the displacement at $t = 1$ s.

Solution:

For critical damping we have $\gamma = c/2m = (k/m)^{1/2} = \omega_0 = 2\pi/T_0$. Hence, $\gamma = 2\pi\ \text{s}^{-1}$ in our case, because $T_0 = 1$ s. Now the general expression for the displacement in the critically damped case $x(t) = (At+B)e^{-\gamma t}$, so, for $t = 0$, $x_0 = B$. Differentiating, we have $\dot{x}(t) = (A - \gamma B - \gamma At)e^{-\gamma t}$, which gives $\dot{x}_0 = A - \gamma B = 0$, so $A = \gamma B = \gamma x_0$ in our problem. Accordingly,

$$x(t) = x_0(1 + \gamma t)e^{-\gamma t} = x_0(1 + 2\pi t)e^{-2\pi t}$$

is the displacement as a function of time. For $t = 1$ s, we obtain

$$x_0(1 + 2\pi)e^{-2\pi} = x_0(7.28)e^{-6.28} = 0.0136\, x_0$$

The system has practically returned to equilibrium.

III. Underdamping case:

If the constant γ is small enough that q is **imaginary. The motion, in this case, is oscillatory but with an ultimate death.** Let introduce the constant ω_d such that; $q = i\omega_d$

Then;

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

Which is known as the **angular frequency of the under-damped oscillator.** The solution for the underdamped oscillator could be;

$$x(t) = e^{-\gamma t}\left(\frac{A}{2}e^{+i(\omega_d t+\theta_0)} + \frac{A}{2}e^{-i(\omega_d t+\theta_0)}\right)$$

We now apply Euler's identity[3] to the above expressions, thus obtaining

$$\frac{A}{2}e^{+i(\omega_d t+\theta_0)} = \frac{A}{2}\cos(\omega_d t + \theta_0) + i\frac{A}{2}\sin(\omega_d t + \theta_0)$$

$$\frac{A}{2}e^{-i(\omega_d t+\theta_0)} = \frac{A}{2}\cos(\omega_d t + \theta_0) - i\frac{A}{2}\sin(\omega_d t + \theta_0)$$

$$\therefore x(t) = e^{-\gamma t}(A\cos(\omega_d t + \theta_0))$$

we can express the solution equally well as a sine function:

$$x(t) = e^{-\gamma t}(A\sin(\omega_d t + \phi_0))$$

The main differences between the solution of the underdamped oscillator and the undamped oscillator are :

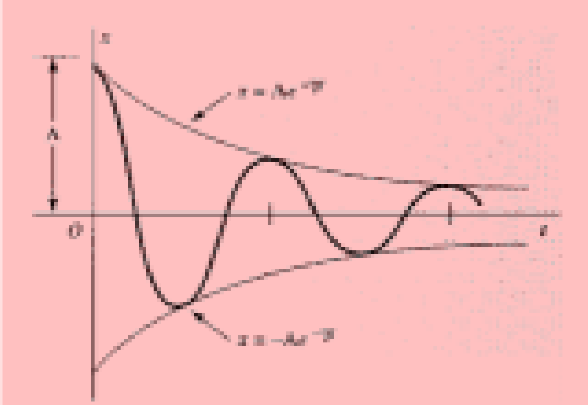
1- The presence of the real exponential factor $e^{-\gamma t}$ leads to a gradual death of the oscillations.

2- The underdamped oscillator vibrates a little more **slowly** than the undamped oscillator does. I.e, $\omega_d < \omega_0$ because of the presence of the damping force.

The period of the underdamped oscillator is given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}}$$

Thus, in **one complete period** the amplitude diminishes by a factor $e^{-\gamma T_d}$



EXAMPLE 3.4.2

The frequency of a damped harmonic oscillator is one-half the frequency of the same oscillator with no damping. Find the ratio of the maxima of successive oscillations.

Solution:

We have $\omega_d = \frac{1}{2}\omega_0 = (\omega_0^2 - \gamma^2)^{1/2}$, which gives $\omega_0^2/4 = \omega_0^2 - \gamma^2$, so $\gamma = \omega_0(3/4)^{1/2}$. Consequently,

$$\gamma T_d = \omega_0(3/4)^{1/2}\,[2\pi/(\omega_0/2)] = 10.88$$

Thus, the amplitude ratio is

$$e^{-\gamma T_d} = e^{-10.88} = 0.00002$$

This is a *highly damped* oscillator.

Energy Considerations

The total energy of the damped harmonic oscillator is given by the sum of the kinetic and potential energies:

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 \qquad (3.4.18)$$

This is constant for the undamped oscillator, as stated previously. Let us differentiate the above expression with respect to t:

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x} \qquad (3.4.19)$$

Now the differential equation of motion is $m\ddot{x} + c\dot{x} + kx = 0$, or $m\ddot{x} + kx = -c\dot{x}$. Thus, we can write

$$\frac{dE}{dt} = -c\dot{x}^2 \tag{3.4.20}$$

Quality Factor

$$Quality\ Factor(Q) = \frac{2\pi\ times\ the\ energy\ stored\ in\ the\ oscillator}{the\ energy\ lost\ in\ a\ single\ period\ of\ oscillation\ T_d} = \frac{2\pi\ E}{(\Delta E)_{T_d}}$$

If the oscillator is weakly damped, the energy lost per cycle is small and Q is large.

The ratio of the energy stored in the oscillator to that lost in a single period of oscillation is characterized by a parameter Q, called **the quality factor**. This factor is related to w_d by the relation

$$Q = \frac{\omega_d}{2\gamma}$$