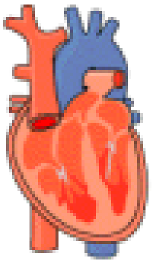
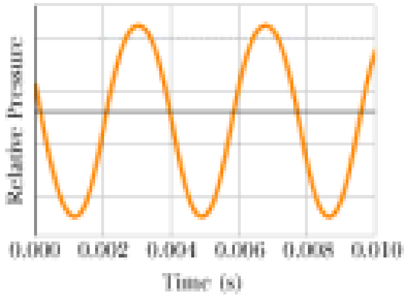
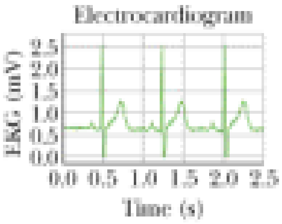
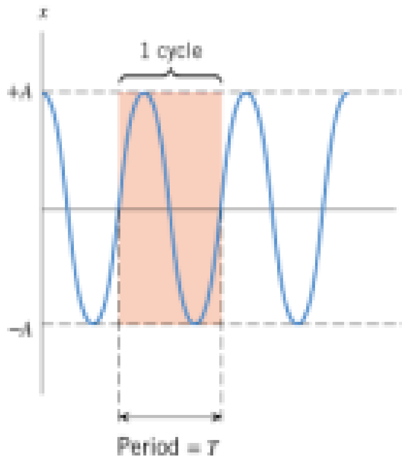
The periodic motion

Any measurable quantity that repeats itself at regular time intervals is defined as undergoing **periodic** motion.
If the periodic variation of a physical quantity over time has the shape of a sine (or cosine) function, we call it a **sinusoidal oscillation or simple harmonic motion.**



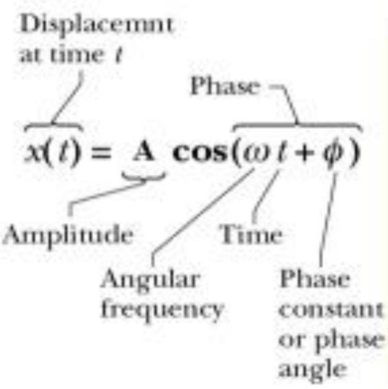






- The **period T** is the time required for one complete motional cycle.
- The **frequency f** of the motion is the number of cycles of the motion per second (unit is: 1 cycle/second=1 Hz).
- Frequency and period are related according to: $f = \frac{1}{T}$

Simple Harmonic Motion

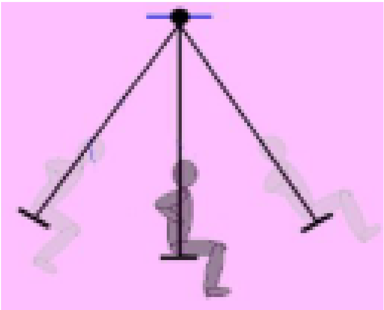


$$T = \frac{2\pi}{\omega}$$

$$T = \frac{1}{f}$$

Note: Angles are in radians.

Introduction to Oscillations



- In order to describe the complicated forms of periodic motion around us, usually we start by an analysis of the simplest form of oscillations,

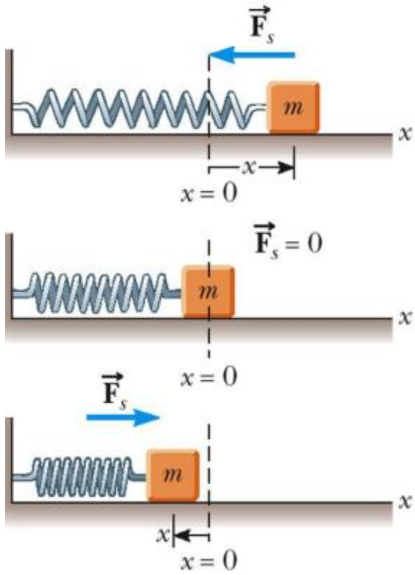
  the simple harmonic motion.

- The main two characteristics of the simple harmonic motion are;

(1) It is described by a second-order, linear differential equation with constant coefficients. Thus, the Superposition principle holds, if two particular solutions are found, their sum is also a solution.

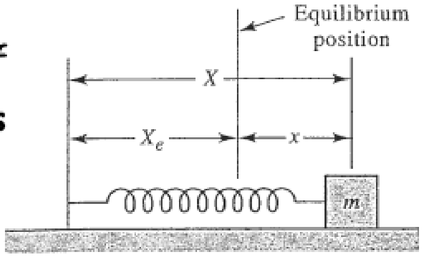
(2) It has Amplitude-independent periods .That is, the periodic time of the motion, is independent of the maximum displacement from equilibrium (the amplitude) .

3.2. Linear Restoring Force: Harmonic Motion



- Consider a mass m on a frictionless surface attached to a wall by means of a spring.

- Let X_e is the unstretched length of the spring. **This position represents the equilibrium position where the potential energy is a minimum.**



- If the mass is pushed or pulled away from this position, the spring will be either compressed or stretched, and then exert a force on the mass.

- This force will always attempt to restore it to its equilibrium position.

- To calculate the motion of the mass, we need an expression for this *restoring force* .

According to *Hooke's law* The spring's restoring force is given by ;

$$F(x) = -kx$$

where k is the spring constant. In fact, this law is valid only for small displacements from equilibrium, where the restoring force is **linear**.

Newton's second law of motion can now be written as

$$m\ddot{x} + kx = 0 \qquad (3.2.4a)$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad (3.2.4b)$$

looking for a solution which can show that **the motion is both periodic and bounded**. *Sine* and *cosine* functions both can exhibit that sort of behavior. Thus, a possible solution is;

$$x = A\sin(\omega_0 t + \phi_0) \qquad (3.2.5)$$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad (3.2.6)$$

is the *angular frequency* of the system.

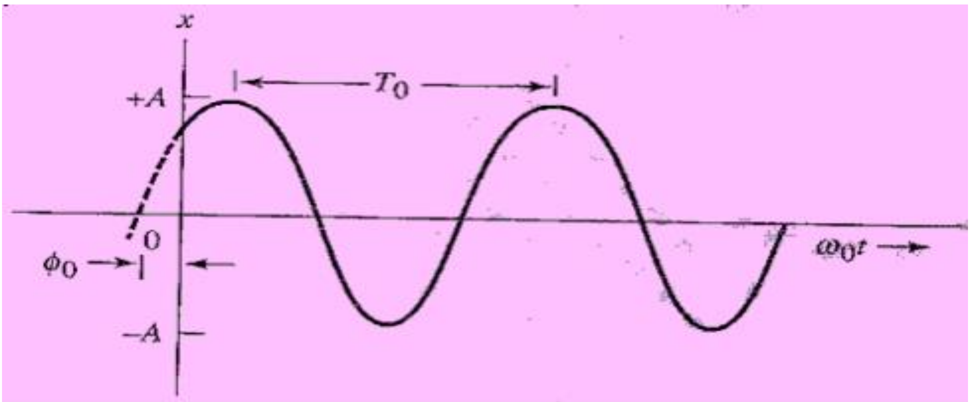


Figure 3.2.3 Displacement versus $\omega_0 t$ for the simple harmonic oscillator.

The motion exhibits the following features:

(1) The motion repeats itself after a time T_0 known as *the period* of the motion. Which is *the time required for a phase advance of 2π*, and is given by

$$\omega_0(t + T_0) + \phi_0 = \omega_0 t + \phi_0 + 2\pi$$

Or;

$$T_0 = \frac{2\pi}{\omega_0}$$

(2) The motion is bounded; that is, it is confined within the limits $-A \le x \le +A$. Where, A is called *the amplitude* of the motion and it is **independent** of ω_0.

(3) The phase angle ϕ_0 is the initial value of the sine function. It determines the value of the displacement x at time $t = 0$. I.e, $x(t=0) = A\sin(\phi_0)$

(4) The term *frequency*, f_0, refer to the reciprocal of the period of the oscillation or

$$f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The unit of frequency (*cycles per second*, or s^{-1}) is called the hertz (Hz).

(5) Constants of the Motion A and ϕ_0, can be determined from the *initial conditions* as follows:

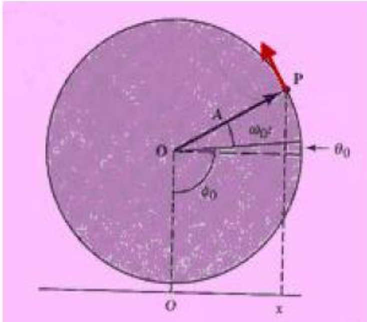
$$x(0) = A\sin(\phi_0) = x_0$$
$$\dot{x}(0) = \omega_0 \; A\cos(\phi_0) = v_0$$
$$\therefore \quad \tan\phi_0 = \frac{\omega_0 x_0}{v_0}$$

$$A^2 = x_0^2 + \frac{v_0^2}{\omega_0^2}$$

Simple Harmonic Motion: The Projection of a Rotating Vector

Imagine a vector **A** rotating at a constant angular velocity w_o . Let this vector denote the position of a point P moves in uniform circular motion.

The **projection of A** traces out simple harmonic motion.



Since $\dot{\theta} = \omega_0$ **and** $\theta = \omega_0 t + \theta_0$

θ_0 is the value of θ at t=0.

Thus, the projection of P onto the x-axis is

$$x = A\cos\theta = A\cos(\omega_0 t + \theta_0)$$

Or with the equivalence expression:

$$x = A\sin(\omega_0 t + \phi_0)$$

$$\cos(\omega_0 t + \theta_0) = \cos\left(\omega_0 t + \phi_0 - \frac{\pi}{2}\right) = \sin(\omega_0 t + \phi_0)$$

where $\phi_0 - \theta_0 = \pi/2$

Or we could represent the general solution for harmonic motion:

$$x = A\sin\phi_0 \cos\omega_0 t + A\cos\phi_0 \sin\omega_0 t = C\cos\omega_0 t + D\sin\omega_0 t$$

Note that: $\tan\phi_0 = \dfrac{C}{D}$, $A^2 = C^2 + D^2$

Effect of a Constant External Force on a Harmonic Oscillator

Suppose the same spring shown in Figure 3.2.1 is held in a vertical position, supporting the same mass m (Fig. 3.2.5). The total force acting is now given by adding the weight mg to the restoring force,

$$F = -k(X - X_e) + mg \qquad (3.2.20)$$

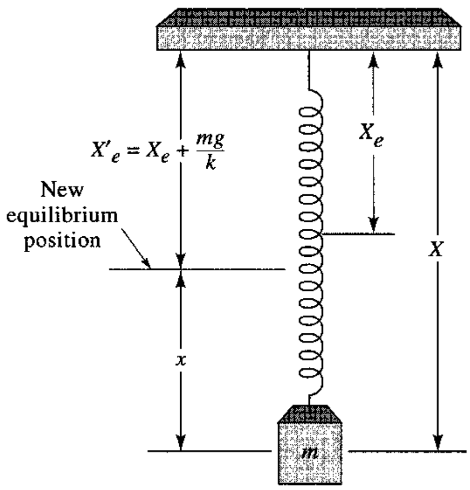
This equation could be written $F = -kx + mg$

where, $x = X - X_e$

let $\acute{X}_e$ be the displacement from the new equilibrium position

obtained by setting $F = 0$

$0 = -k(X_e' - X_e) + mg$, which gives $X_e' = X_e + mg/k$.



We now define the displacement as

$$x = X - X_e' = X - X_e - \frac{mg}{k}$$

$$F = -kx \quad (3.2.22)$$

so the differential equation of motion is again

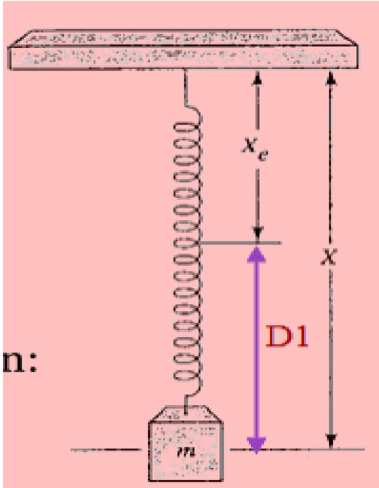
$$m\ddot{x} + kx = 0 \quad (3.2.23)$$

and our solution in terms of our newly defined x is identical to that of the horizontal case.

Example (3.2.1): Effect of a Constant External Force

When a light spring supports a block of mass m vertically, the spring is found to stretch by an amount D1 over its unstreched length. If the block is furthermore pulled downward a distance D2 then released at time $t = 0$, find:

(a) the resulting motion.
(b) the velocity of the block at the equilibrium position.
(c) the acceleration of the block at the top of its oscillatory motion.



n:

Solution:

Assuming the positive direction is down, at the equilibrium position we have

$$F_x = 0 = -kD_1 + mg$$

This gives us the value of the spring constant k:

$$k = \frac{mg}{D_1}$$

From this we can find the angular frequency of oscillation:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{D_1}}$$

We will express the motion in the form

Then,
$$x = A\cos\omega_0 t + B\sin\omega_0 t$$

$$\dot{x} = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t$$

Applying the initial conditions , we find

$x_0 = D_2 = A$ & $v_0 = 0 = B\omega_0 \Longrightarrow B = 0$

The motion is, therefore, given by

(a) $$x = D_2 \cos\left(\sqrt{\frac{g}{D_1}}t\right)$$

Note that the mass m does not appear in the final expression.
The velocity is then

(b) $$\dot{x} = -D_2\sqrt{\frac{g}{D_1}}\sin\left(\sqrt{\frac{g}{D_1}}t\right) \xrightarrow{\text{(center)}} \dot{x} = -D_2\sqrt{\frac{g}{D_1}}$$

and the acceleration

(c) $$\ddot{x} = -D_2\frac{g}{D_1}\cos\left(\sqrt{\frac{g}{D_1}}t\right) \xrightarrow{\text{(top)}} \ddot{x} = D_2\frac{g}{D_1}$$

In the case $D_1 = D_2$ the downward acceleration at the top of the swing is just g.
This means that the block, at that particular instant, is in free fall; that is, the spring is exerting zero force on the block.

Example (3.2.2):

The simple pendulum consists of a small mass m swinging at the end of a light string of length l. The motion is along a circular arc defined by the angle θ, as shown in the Fig.

The restoring force is ; $F_s = -mg \sin\theta$.
Therefore, the differential equation of motion is;

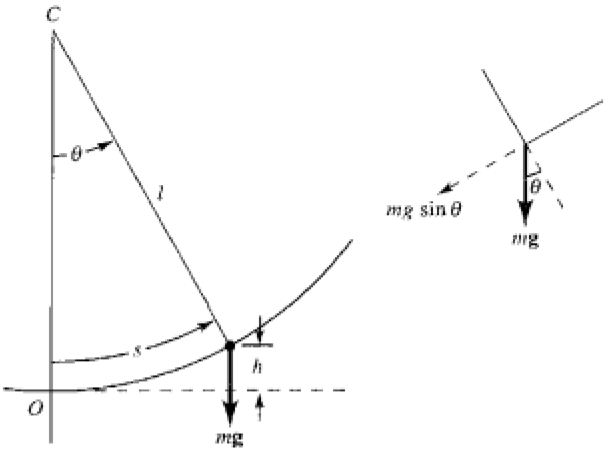
$$m\ddot{s} = -mg\sin\theta$$

Since $s = l\theta$ and, for small θ, $\sin\theta = \theta$, we can write the differential equation of motion as follows:

or

$$\ddot{s} + \frac{g}{l} s = 0$$

$$\ddot{\theta} + \frac{g}{l} \theta = 0$$



Although the motion is along a curved path, the differential equation is mathematically identical to that of the linear harmonic oscillator;

$$\ddot{x} + \frac{k}{m} x = 0$$

Thus, for the angles that the approximation sin θ = θ is valid, we can conclude that the motion is simple harmonic with angular frequency

$$\theta = \theta_0 \cos(\omega_0 t + \phi_0)$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

and period

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$$

3.3 Energy Considerations in Harmonic Motion

Consider a particle under the action of a linear restoring force $F_x = -kx$. Let us calculate **the work** done by an external force F_{ext} in moving the particle from the equilibrium position ($x = 0$) to some position x. We have, $F_{ext} = -F_x = kx$, so

$$W = \int_0^x F_{ext} dx = \int_0^x kx\, dx = \frac{k}{2}x^2$$

This work is stored in the spring as potential energy: $V(x)$, where

$$V(x) = \tfrac{1}{2}kx^2$$

Thus, $F_x = -dV/dx = -kx$, as required by the definition of V. The total energy, when the particle is undergoing harmonic motion, is given by the sum of the kinetic and potential energies, namely,

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 \tag{3.3.3}$$

The motion of the particle can be found by starting with the energy equation (3.3.3). Solving for the velocity gives

$$\dot{x} = \pm\left(\frac{2E}{m} - \frac{kx^2}{m}\right)^{1/2} \tag{3.3.4}$$

which can be integrated to give t as a function of x as follows:

$$t = \int \frac{dx}{\pm[(2E/m) - (k/m)x^2]^{1/2}} = \mp(m/k)^{1/2}\cos^{-1}(x/A) + C \tag{3.3.5}$$

in which C is a constant of integration and A is the amplitude given by

$$A = \left(\frac{2E}{k}\right)^{1/2} \tag{3.3.6}$$

We also see from the energy equation that the maximum value of the speed, which we call v_{max}, occurs at $x = 0$. Accordingly, we can write

$$E = \tfrac{1}{2}mv_{max}^2 = \tfrac{1}{2}kA^2 \tag{3.3.7}$$

As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of kinetic energy at the center, where $x = 0$ and $\dot{x} = \pm v_{max}$, and it is all potential energy at the extrema, where $\dot{x} = 0$ and $x = \pm A$.

EXAMPLE 3.3.1

The Energy Function of the Simple Pendulum

The potential energy of the simple pendulum (Fig. 3.2.6) is given by the expression

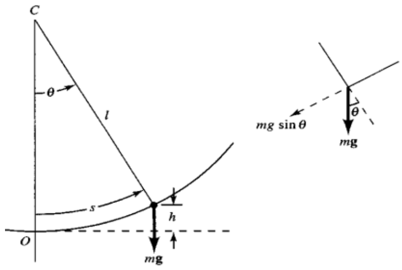
$$V = mgh$$

where h is the vertical distance from the reference level (which we choose to be the level of the equilibrium position). For a displacement through an angle θ (Fig. 3.2.6), we see that $h = l - l\cos\theta$, so

$$V(\theta) = mgl(1 - \cos\theta)$$

Now the series expansion for the cosine is $\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \cdots$, so for small θ we have approximately $\cos\theta = 1 - \theta^2/2$. This gives

$$V(\theta) = \tfrac{1}{2} mgl\,\theta^2$$



or, equivalently, because $s = l\theta$,

$$V(s) = \tfrac{1}{2}\frac{mg}{l}s^2$$

Thus, to a first approximation, the potential energy function is quadratic in the displacement variable. In terms of s, the total energy is given by

$$E = \tfrac{1}{2}m\dot{s}^2 + \tfrac{1}{2}\frac{mg}{l}s^2$$

in accordance with the general statement concerning the energy of the harmonic oscillator discussed above.

EXAMPLE 3.3.2

Calculate the average kinetic, potential, and total energies of the harmonic oscillator. (Here we use the symbol K for kinetic energy and T_0 for the period of the motion.)

Solution:

$$\langle K\rangle = \frac{1}{T_0}\int_0^{T_0} K(t)\,dt = \frac{1}{T_0}\int_0^{T_0}\tfrac{1}{2}m\dot{x}^2\,dt$$

but

$$x = A\sin(\omega_0 t + \phi_0)$$
$$\dot{x} = \omega_0 A\cos(\omega_0 t + \phi_0)$$

Setting $\phi_0 = 0$ and letting $u = \omega_0 t = (2\pi/T_0)\cdot t$, we obtain

$$\langle K\rangle = \frac{1}{T_0}\left[\tfrac{1}{2}m\omega_0^2 A^2\int_0^{T_0}\cos^2(\omega_0 t)\,dt\right]$$
$$= \frac{1}{2\pi}\left[\tfrac{1}{2}m\omega_0^2 A^2\int_0^{2\pi}\cos^2 u\,du\right]$$

We can make use of the fact that

$$\frac{1}{2\pi}\int_0^{2\pi}(\sin^2 u + \cos^2 u)\,du = \frac{1}{2\pi}\int_0^{2\pi} du = 1$$

to obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^2 u\,du = \tfrac{1}{2}$$

because the areas under the $\cos^2$ and $\sin^2$ terms throughout one cycle are identical. Thus,

$$\langle K\rangle = \tfrac{1}{4} m\omega_0^2 A^2$$

The calculation of the average potential energy proceeds along similar lines.

$$V = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\ \sin^2\omega_0 t$$

$$\langle V\rangle = \tfrac{1}{2}kA^2\,\frac{1}{T_0}\int_0^{T_0}\ \sin^2\omega_0 t\,dt$$

$$= \tfrac{1}{2}kA^2\,\frac{1}{2\pi}\int_0^{2\pi}\ \sin^2 u\,du$$

$$= \tfrac{1}{4}kA^2$$

Now, because $k/m = \omega_0^2$ or $k = m\omega_0^2$, we obtain

$$\langle V\rangle = \tfrac{1}{4}kA^2 = \tfrac{1}{4}m\omega_0^2A^2 = \langle K\rangle$$

$$\langle E\rangle = \langle K\rangle + \langle V\rangle = \tfrac{1}{2}m\omega_0^2A^2 = \tfrac{1}{2}kA^2 = E$$

The average kinetic energies and potential energies are equal; therefore, the average energy of the oscillator is equal to its total instantaneous energy.