2

Newtonian Mechanics:

Rectilinear Motion of a Particle

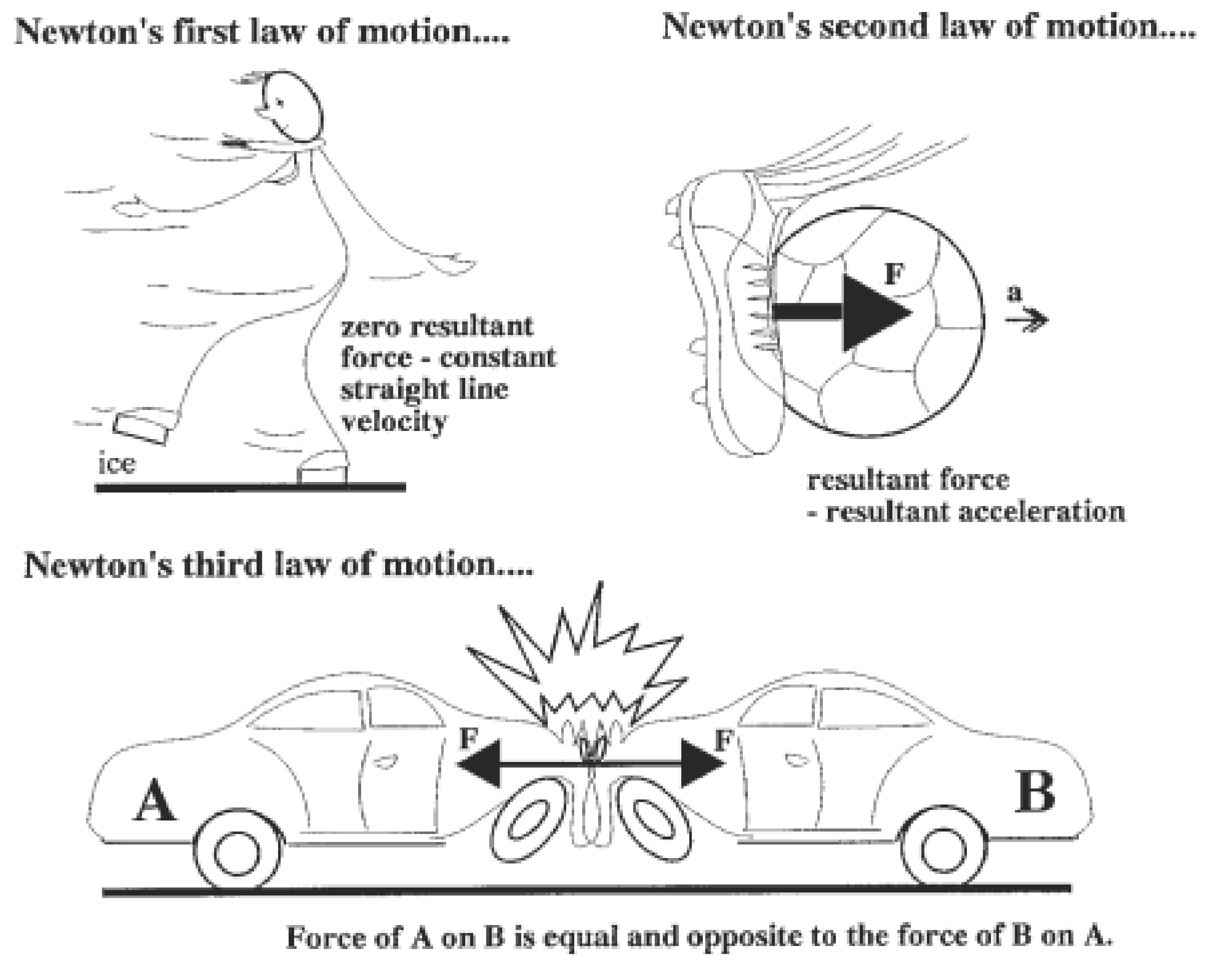
2.1 Newton's Laws of Motion: Historical Introduction

In his *Principia* of 1687, Isaac Newton laid down three fundamental laws of motion, which would forever change mankind's perception of the world:

I. Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

II. The change of motion is proportional to the motive force impressed and is made in the direction of the line in which that force is impressed.

III. To every action there is always imposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.



Newton’s First Law: Inertial Reference Systems

The first law describes a common property of matter, known as **inertia.**

- The physical quantity that measures inertia is called **mass**.

What is Inertia?

It is the resistance of a matter to change its state of motion. This means that; in the absence of applied forces, matter simply persists in its current velocity state-forever.

Inertial Frames of Reference:

- A mathematical description of the motion of a particle requires the selection of a frame of reference, or a set of coordinates according to which the position, velocity, and acceleration of the particle can be specified.

- Uniformly moving reference frames (e.g. those considered at 'rest' or moving with constant velocity in a straight line) are called **inertial reference frames**.

- If we can neglect the effect of the earth’s rotations, a frame of reference fixed in the earth is an inertial reference frame.

- Newton’s laws are only applicable at inertial reference frames.

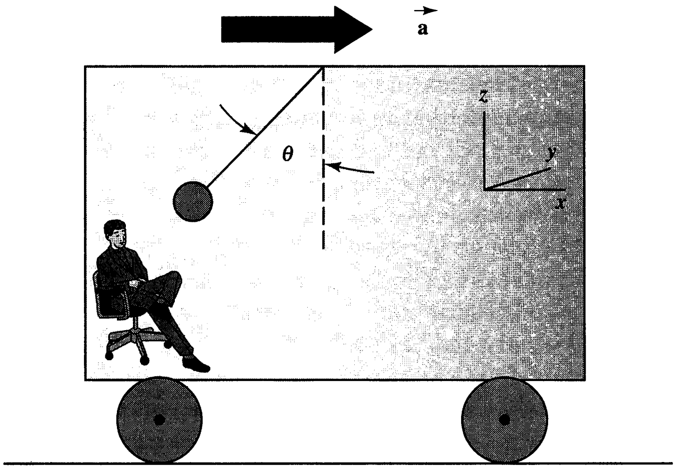


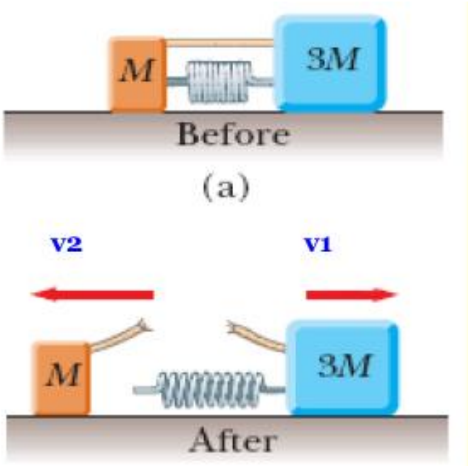
Figure 2.1.1 A plumb bob hangs at an angle θ in an accelerating frame of reference.

A simple example of a noninertial frame of reference

Newton's Second and Third Laws (Mass and Force):

- The more massive an object is, the more resistive it is to acceleration.

Suppose we have two masses m1 , m2 on a frictionless surface. Now imagine someone pushing the two masses together, and then suddenly releasing them so that they fly apart, achieving speeds v1 and v2 .



The ratio of the two masses can be expressed as;

$$\frac{m_2}{m_1} = \left|\frac{\mathbf{v}_1}{\mathbf{v}_2}\right| \qquad (2.1.1)$$

Or $$\Delta(m_1\mathbf{v}_1) = -\Delta(m_2\mathbf{v}_2) \qquad (2.1.2)$$

The (-) appears because the final velocities v1 and v2 are in opposite directions. If we divide by Δt and take limits as $\Delta t \to 0$ we obtain,

$$\frac{d}{dt}(m_1\mathbf{v}_1) = -\frac{d}{dt}(m_2\mathbf{v}_2) \qquad (2.1.3)$$

According to Newton's 2nd law, this “change of motion” is proportional to the force caused it;

$$\mathbf{F} \propto \frac{d}{dt}(m\mathbf{v})$$

Defining the unit in the SI system, Newton's 2nd law can be expressed in the familiar form:

Thus, we finally express Newton’s second law in the familiar form

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} = m\mathbf{a} \qquad (2.1.6)$$

The force **F** on the left side of Equation (2.1.6) is the net force $\mathbf{F}_{\mathbf{net}}$ acting upon the mass **m**.

We note that Equation 2.1.3 is equivalent to

$$\mathbf{F}_1 = -\mathbf{F}_2 \qquad (2.1.7)$$

Which is Newton's 3rd law, that states; two interacting bodies exert equal and opposite forces upon one another.

Linear Momentum

Since, $\mathbf{p} = m\mathbf{v}$ is called linear momentum, the 2nd law can be rewritten as;

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad (2.1.9)$$

Thus, Equation 2.1.3, which describes the behavior of two mutually interacting masses, is equivalent to

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0 \qquad (2.1.10)$$

or

$$\mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \qquad (2.1.11)$$

In other words, Newton's 3rd law implies that the total momentum of two mutually interacting bodies is a constant.

EXAMPLE 2.1.2

A spaceship of mass M is traveling in deep space with velocity $v_i = 20$ km/s relative to the Sun. It ejects a rear stage of mass $0.2\,M$ with a relative speed $u = 5$ km/s (Figure 2.1.4). What then is the velocity of the spaceship?

Solution:

The system of spaceship plus rear stage is a closed system upon which no external forces act (neglecting the gravitational force of the Sun); therefore, the total linear momentum is conserved. Thus

$$\mathbf{P}_f = \mathbf{P}_i$$

where the subscripts i and f refer to initial and final values respectively. Taking velocities in the direction of the spaceship's travel to be positive, before ejection of the rear stage, we have

$$P_i = Mv_i$$

Let U be the velocity of the ejected rear stage and v_f be the velocity of the ship after ejection. The total momentum of the system after ejection is then

$$P_f = 0.20\,MU + 0.80\,Mv_f$$

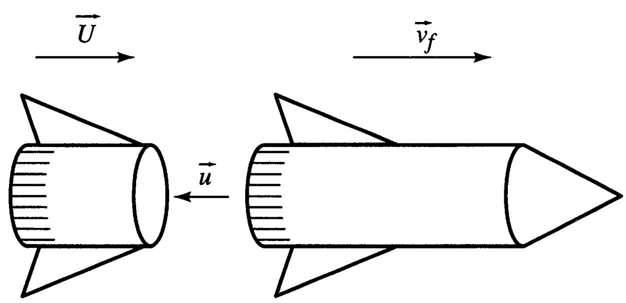


Figure 2.1.4 Spaceship ejecting a rear stage.

The speed u of the ejected stage relative to the spaceship is the difference in velocities of the spaceship and stage

$$u = v_f - U$$

or

$$U = v_f - u$$

Substituting this latter expression into the equation above and using the conservation of momentum condition, we find

$$0.20\, M(v_f - u) + 0.8\, Mv_f = Mv_i$$

which gives us

$$v_f = v_i + 0.2u = 20 \text{ km/s} + 0.20\ (5 \text{ km/s}) = 21 \text{ km/s}$$

Motion of a Particle

The fundamental equation of motion for a particle subject to the influence of a net force, **F** is,

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} = m\mathbf{a}$$

By writing **F** as $\mathbf{F}_{\mathbf{net}}$ the vector sum of all the forces acting on the particle.

$$\mathbf{F}_{\mathbf{net}} = \sum \mathbf{F}_i = m\frac{d^2\mathbf{r}}{dt^2} = m\mathbf{a} \qquad (2.1.12)$$

The usual problem of dynamics can be expressed in the following way:

1- Given a knowledge of the forces acting on a particle (or system of particles)
2- Calculate the acceleration of the particle.
3- Calculate the velocity and position as functions of time.

This process involves solving the second-order differential equation of motion, by using the initial condition of the problem, such as the values of the position and velocity of the particle at time t=0.

2.2 Rectilinear Motion: Uniform Acceleration Under a Constant Force

When a moving particle remains on a single straight line, the motion is said to be rectilinear. In this case, we can choose the x-axis as the line of motion, The general equation of motion is then,

$$F_x(x, \dot{x}, t) = m\ddot{x} \qquad (2.2.1)$$

1- The simplest case is when **F** is constant.

In this case **a** is constant ;

$$\ddot{x} = \frac{dv}{dt} = \frac{F}{m} = \text{constant} = a \qquad (2.2.2a)$$

and the solution is obtained by direct integration with respect to time:

$$\dot{x} = v = at + v_0 \qquad (2.2.2b)$$

$$x = \tfrac{1}{2}at^2 + v_0 t + x_0 \qquad (2.2.2c)$$

where v_0 is the velocity and x_0 is the position at $t = 0$. from the above equations we obtain

$$2a(x - x_0) = v^2 - v_0^2$$

There are a number of fundamental applications for such case. For example, the acceleration of a body falling freely near the surface of the Earth, neglecting air resistance, is nearly constant.

EXAMPLE 2.2.1

Consider a block that is free to slide down a smooth, frictionless plane that is inclined at an angle θ to the horizontal, as shown in Figure 2.2.1(a). If the height of the plane is h and the block is released from rest at the top, what will be its speed when it reaches the bottom?

Solution:

We choose a coordinate system whose positive x-axis points down the plane and whose y-axis points "upward," perpendicular to the plane, as shown in the figure. The only force along the x direction is the component of gravitational force, $mg \sin\theta$, as shown in Figure 2.2.1(b). It is constant. Thus, Equations 2.2.2a–d are the equations of motion, where

$$\ddot{x} = a = \frac{F_x}{m} = g \sin\theta$$

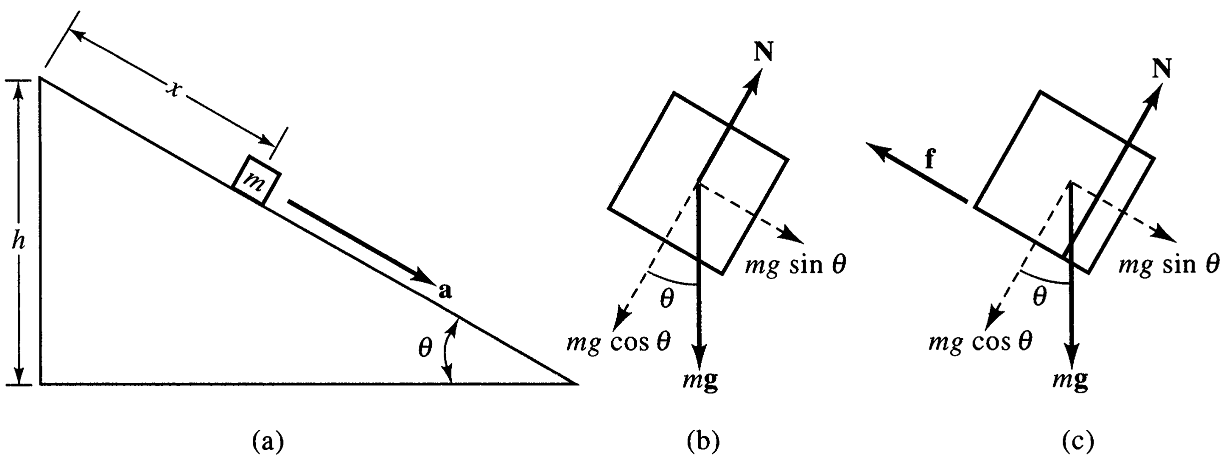


Figure 2.2.1 (a) A block sliding down an inclined plane. (b) Force diagram (no friction). (c) Force diagram (friction $f = \mu_K \mathbf{N}$).

and

$$x - x_0 = \frac{h}{\sin\theta}$$

Thus,

$$v^2 = 2(g\ \sin\theta)\left(\frac{h}{\sin\theta}\right) = 2gh$$

Suppose that, instead of being smooth, the plane is rough; that is, it exerts a frictional force **f** on the particle. Then the net force in the x direction, (see Figure 2.2.1(c)), is equal to $mg\ \sin\theta - f$. Now, for sliding contact it is found that the magnitude of the frictional force is proportional to the magnitude of the normal force N; that is,

$$f = \mu_K N$$

where the constant of proportionality μ_K is known as the *coefficient of sliding* or *kinetic friction.*[7] In the example under discussion, the normal force, as shown in the figure, is equal to $mg\ \cos\theta$; hence,

$$f = \mu_K mg\ \cos\theta$$

Consequently, the net force in the x direction is equal to

$$mg\ \sin\theta - \mu_K mg\ \cos\theta$$

Again the force is constant, and Equations 2.2.2a–d apply where

$$\ddot{x} = \frac{F_x}{m} = g(\sin\theta - \mu_k\ \cos\theta)$$

The angle, $\tan^{-1}\mu_K$, usually denoted by ϵ, is called the *angle of kinetic friction.* The speed of the particle increases if $\theta > \tan^{-1}\mu_K$.

the particle slides down the plane with constant speed. If $\theta = \epsilon$, then $a = 0$,

If $\theta < \epsilon$, a is negative, and so the particle eventually comes to rest.

2.3 Forces that Depend on Position: The Concepts of Kinetic and Potential Energy

If the force is independent of velocity or time, then the differential equation for rectilinear motion is simply

$$F(x) = m\ddot{x} \qquad (2.3.1)$$

using the chain rule to write the acceleration in the following way:

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = v\frac{dv}{dx} \qquad (2.3.2)$$

so the differential equation of motion may be written

$$F(x) = mv\frac{dv}{dx} = \frac{m}{2}\frac{d(v^2)}{dx} = \frac{dT}{dx} \qquad (2.3.3)$$

By integrating equation (2.3.3) we get the **kinetic energy** of the particle, $T = \frac{1}{2}mv^2$

$$W = \int_{x_0}^{x} F(x)\,dx = T - T_0 \qquad (2.3.4)$$

The integral $\int F(x)dx$ is the *work* W done on the particle by the impressed force $F(x)$.

Let us define a function **V(x)** such that,

$$-\frac{dV(x)}{dx} = F(x) \qquad (2.3.5)$$

The function **V(x)** is called the **potential energy**

In terms of $V(x)$, the work integral is

$$W = \int_{x_0}^{x} F(x)\,dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0 \qquad (2.3.6)$$

From (2.3.6);

$$T_o + V(x_o) = T + V(x) \equiv E \qquad \textbf{(2.3.8)}$$

E is known as **total mechanical energy** of particle.

Note:

1- E the sum of the kinetic and potential energies is constant throughout the motion of the particle.

2- The force is a function only of the position x. Such a force is said to be **conservative**.

3- $v=0$ when $V(x)=E$. This point known as "the turning point"

The motion of the particle can be obtained by solving the energy equation (Equation 2.3.8) for v,

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]} \tag{2.3.9}$$

which can be written in integral form,

$$\int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{2}{m}[E - V(x)]}} = t - t_0 \tag{2.3.10}$$

thus giving t as a function of x.

Exp (2.3.1): Free Fall

The motion of a freely falling body is an example of conservative motion. In this case:

Hence;

$$F = -\frac{dV}{dx} = -mg$$

$$V = mgx + C$$

We can choose $C = 0$, which means that $V = 0$ when $x = 0$. The energy equation is then

$$\tfrac{1}{2}mv^2 + mgx = E$$

For instance, let the body be projected upward with initial speed v_0 from the origin $x=0$. These values give;

$$\tfrac{1}{2}mv^2 + mgx = \tfrac{1}{2}mv_0^2 + 0$$

so;

$$v^2 = v_0^2 - 2gx$$

The turning point of the motion, which is in this case the maximum height, is given by setting $v = 0$. This gives

$$h = x_{max} = \frac{v_0^2}{2g}$$

EXAMPLE 2.3.2

Variation of Gravity with Height

(Newton's law of gravity).[9] Thus, the gravitational force that the Earth exerts on a body of mass m is given by

$$F_r = -\frac{GMm}{r^2}$$

in which **G** is Newton's constant of gravitation, **M** is the mass of the Earth, and **r** is the distance from the center of the Earth to the body.

When the body is at the surface of the Earth,

$$\boldsymbol{F_r = -mg} \quad \text{and} \quad \boldsymbol{mg = \frac{GMm}{r_e^2}} \quad \text{thus} \quad \boldsymbol{g = \frac{GM}{r_e^2}}$$

Let x be the distance above the surface, so that, $r = r_e + x$,

$$F(x) = -mg\frac{r_e^2}{(r_e+x)^2} = m\ddot{x}$$

To integrate, we set $\ddot{x} = v dv/dx$. Then

$$-mgr_e^2\int_{x_0}^{x}\frac{dx}{(r_e+x)^2} = \int_{v_0}^{v} mv\,dv$$

$$mgr_e^2\left(\frac{1}{r_e+x} - \frac{1}{r_e+x_0}\right) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2$$

From the equation,

$$W = \int_{x_0}^{x} F(x)\,dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0$$

The potential energy is,

$$V(x) = -\frac{mgr_e}{r_e + x}$$

Maximum Height: Escape Speed

Suppose a body is projected upward with initial speed v_0 at the surface of the Earth,

The energy equation then yields,

$$mgr_e^2\left(\frac{1}{r_e+x}-\frac{1}{r_e+x_0}\right)=\frac{1}{2}mv^2-\frac{1}{2}mv_0^2$$

Let $x_o=0$ then solving for v^2 ,

$$2gr_e^2\left(\frac{1}{r_e+x}-\frac{1}{r_e}\right)=v^2-v_o^2$$

$$2gr_e^2\left(\frac{-x}{(r_e+x)r_e}\right)=v^2-v_o^2$$

$$2g\left(\frac{-x}{\frac{r_e+x}{r_e}}\right)=v^2-v_o^2$$

$$-2gx\left(\frac{1}{1+\frac{x}{r_e}}\right)=v^2-v_o^2$$

$$v^2=v_0^2-2gx\left(1+\frac{x}{r_e}\right)^{-1}$$

The turning point (maximum height) is found by setting $v = 0$ and solving for x. The result is,

$$x_{max}=h=\frac{v_0^2}{2g}\left(1-\frac{v_0^2}{2gr_e}\right)^{-1}$$

(H.W)

Using this last, exact expression, we solve for the value of v_0 that gives an infinite value for h. This is called the *escape speed,* and it is found by setting the quantity in parentheses equal to zero. The result is

$$v_e=(2gr_e)^{1/2}$$

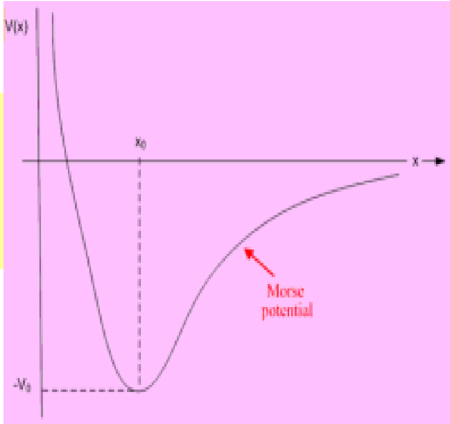
This gives, for $g = 9.8\ \text{m/s}^2$ and $r_e = 6.4\times10^6$ m,

$$v_e\simeq 11\ \text{km/s}\simeq 7\ \text{mi/s}$$

EXAMPLE 2.3.3

The *Morse function* $V(x)$ approximates the potential energy of a vibrating diatomic molecule as a function of x, the distance of separation of its constituent atoms, and is given by

$$V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$$



Show that:

1- x_0 is the separation of the two atoms at equilibrium, i.e. when the *potential energy function* is **minimum**.

2- and that $V(x_0) = -V_0$.

Solution $V(x)$ is min when its derivative (w.r.t) x is zero;

$$F(x) = -\frac{dV(x)}{dx} = 0$$

$$2\frac{V_0}{\delta}\left(1 - e^{-(x-x_0)/\delta}\right)\left(e^{-(x-x_0)/\delta}\right) = 0$$

$$1 - e^{-(x-x_0)/\delta} = 0$$

$$\ln(1) = -(x - x_0)/\delta$$

$$\therefore \quad x = x_0$$

Substituting in the main equation, the value of the min $V(x)$ can be found as; $V(x_0) = -V_0$

EXAMPLE 2.3.4

Shown in Figure 2.3.2 is the potential energy function for a diatomic molecule.

Solution:

All we need do here is expand the potential energy function near the equilibrium position.

$$V(x) \approx V_0\left[1 - \left(1 - \left(\frac{x - x_0}{\delta}\right)\right)\right]^2 - V_0$$

$$\approx \frac{V_0}{\delta^2}(x - x_0)^2 - V_0$$

$$F(x) = -\frac{dV(x)}{dx} = -\frac{2V_0}{\delta^2}(x - x_0)$$

2.4 Velocity-Dependent Forces: Fluid Resistance and Terminal Velocity

It often happens that the force that acts on a body is a function of the velocity of the body. For example, in the case of viscous resistance exerted on a body moving through a fluid. If the force can be expressed as a function of v only, the differential equation of motion may be written in either of the two forms;

$$F_0 + F(v) = m\frac{dv}{dt} \qquad (2.4.1)$$

$$F_0 + F(v) = mv\frac{dv}{dx} \qquad (2.4.2)$$

The general solution for these equations are,

$$\int \frac{dt}{m} = \int \frac{dv}{F_o + F(v)}$$

$$\int \frac{dx}{m} = \int \frac{vdv}{F_o + F(v)}$$

Here F_0 is any constant force that does not depend on v.

F(v) is not a simple function and generally must be found through experimental measurements. However, approximation for many cases is given by the equation,

$$F(v) = -c_1 v - c_2 v\,|v| = -v\,(c_1 + c_2\,|v|) \qquad (2.4.3)$$

in which c_1 and c_2 are constants whose values depend on the size and shape of the body.

Note, (The absolute-value sign is necessary on the last term because the force of fluid resistance is always opposite to the direction of v.)

1-For **small v** the **linear term (c_1v)** in F(v) can be used , while the

2- For **large v** the **quadratic term($c_2 v|v|$)** dominates .

Linear or Quadratic ?

the ratio of, $\frac{F_{quad}}{F_{lin}}$

$$\frac{f_{quad}}{f_{lin}} = \frac{c_2 v|v|}{c_1 v}$$

If the value of v will make the ratio ($\frac{F_{quad}}{F_{lin}} \approx 1$)then it is a **quadratic** case, **otherwise**, it is a **linear** one.

Examples

For spheres in air,

$c_1 = 1.55 \times 10^{-4} D$ & $c_2 = 0.22\, D^2$

where D is the diameter of the sphere in meters.

$$\frac{f_{quad}}{f_{lin}} = \frac{c_2 v|v|}{c_1 v} = \frac{0.22 v|v| D^2}{1.55\times10^{-4} vD} = \left(1.4\times10^{3}\right)|v| D$$

For small v the linear term in F(v) can be used , while the quadratic term dominates at large v.

- A Baseball and Some Drops of Liquid
  - Assess the relative importance of the linear and quadratic drag forces on a baseball of diameter D = 7 cm , traveling at a modest v = 5 m/s. Do the same for a drop of rain (D = 1 mm and v = 0.6 m/s) and for a tiny droplet of oil used in the Millikan oildrop experiment (D = 1.5 μm and $v = 5\text{x}10^{-5}$ m/s).
  - Baseball

$$\frac{f_{quad}}{f_{lin}} = \left(1.4\times10^{3}\right)(0.07)(5) \approx 500$$

shows that f_{lin} is completely negligible for a baseball. Use $\mathbf{f} = -c_2 v|v|$

  - Rain

$$\frac{f_{quad}}{f_{lin}} = \left(1.4\times10^{3}\right)(0.001)(0.6) \approx 1$$

shows that both are needed. Must use full expression $\mathbf{f} = -(c_1 v + c_2 v|v|)$

  - Oil Drop

$$\frac{f_{quad}}{f_{lin}} = \left(1.4\times10^{3}\right)(1.5\times10^{-6})(5\times10^{-5}) \approx 10^{-7}$$

shows that f_{quad} is completely negligible for the oil drop. Use $\mathbf{f} = -c_1 v$

EXAMPLE 2.4.1

Horizontal Motion with Linear Resistance

Suppose a block is projected with initial velocity v_0 on a smooth horizontal surface and that there is air resistance such that the linear term dominates. Then, in the direction of the motion, $F_0 = 0$ in Equations 2.4.1 and 2.4.2, and $F(v) = -c_1 v$. The differential equation of motion is then

$$-c_1 v = m\frac{dv}{dt}$$

which gives, upon integrating,

$$t = \int_{v_0}^{v} -\frac{m\,dv}{c_1 v} = -\frac{m}{c_1}\ln\left(\frac{v}{v_0}\right)$$

Solution:

We can easily solve for v as a function of t by multiplying by $-c_1/m$ and taking the exponential of both sides. The result is

$$v = v_0 e^{-c_1 t/m}$$

Thus, the velocity decreases exponentially with time. A second integration gives

$$x = \int_0^t v_0 e^{-c_1 t/m} dt$$
$$= \frac{mv_0}{c_1}(1 - e^{-c_1 t/m})$$

showing that the block approaches a limiting position given by $x_{lim} = mv_0/c_1$.

EXAMPLE 2.4.2

Horizontal Motion with Quadratic Resistance

If the parameters are such that the quadratic term dominates, then for positive v we can write

$$-c_2 v^2 = m\frac{dv}{dt}$$

which gives

$$t = \int_{v_n}^{v} \frac{-m dv}{c_2 v^2} = \frac{m}{c_2}\left(\frac{1}{v} - \frac{1}{v_0}\right)$$

Solution:

Solving for v, we get

$$v = \frac{v_0}{1 + kt}$$

where $k = c_2 v_0/m$. A second integration gives us the position as a function of time:

$$x(t) = \int_0^t \frac{v_0 dt}{1 + kt} = \frac{v_0}{k}\ln(1 + kt)$$

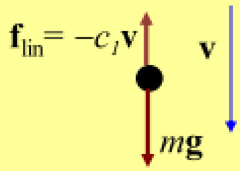
Vertical Fall Through a Fluid: Terminal Velocity

1- Linear Case Vertical Fall through a Fluid

For an object falling vertically in a resisting fluid, the force F_0 in this case, is the weight of the object, $-mg$. For the linear case of fluid resistance, the differential equation of motion is;

$$-mg - c_1 v = m\frac{dv}{dt}$$

Integrating and solving for v, we get $v = -\frac{mg}{c_1} + (\frac{mg}{c_1} + v_0)e^{-c_1 t/m}$



Terminal Velocity

After a sufficient time ($t >> m/c_1$), the velocity approaches a limiting value ($-mg/c_1$). This limiting velocity of a falling body is called ***the terminal velocity*** *(v_t)*. Hence *the terminal speed* is; $v_t = \frac{mg}{c_1}$

The value of v_t/g *is known as* the ***characteristic time*** of the motion (τ). i.e , $\tau = \frac{v_t}{g} = \frac{m}{c_1}$

Looking at the dependence of the terminal speed, you can see that a more massive object has a larger terminal speed. Conversely, if air resistance is great (value of c_1 is large), the terminal speed is small.

At the velocity v_t the force of air resistance is just equal and opposite to the weight of the body (c_1**v =-mg**) so that the **net force** is **zero**, and so the **acceleration** is **zero**.

2- Quadratic case:

In this case $F(v) \propto v^2$ and the *differential equation* of motion is;

$$-mg - c_2 v^2 = m\frac{dv}{dt}$$

Similarly, ***the terminal speed*** is ;

And the ***characteristic time*** is; $\tau = \frac{v_t}{g} = \sqrt{\frac{m}{c_2 g}}$