1.8 Change of Coordinate System: The Transformation Matrix

In this section we show how to represent a vector in different coordinate systems. Consider the vector **A** expressed relative to the triad **ijk**:

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z \tag{1.8.1}$$

Relative to a new triad **i′j′k′** having a different orientation from that of **ijk**, the same vector **A** is expressed as

$$\mathbf{A} = \mathbf{i}'A_{x'} + \mathbf{j}'A_{y'} + \mathbf{k}'A_{z'} \tag{1.8.2}$$

Now the dot product $\mathbf{A} \cdot \mathbf{i}'$ is just $A_{x'}$, that is, the projection of **A** on the unit vector **i′**. Thus, we may write

$$\begin{aligned} A_{x'} &= \mathbf{A}\cdot\mathbf{i}' = (\mathbf{i}\cdot\mathbf{i}')A_x + (\mathbf{j}\cdot\mathbf{i}')A_y + (\mathbf{k}\cdot\mathbf{i}')A_z \\ A_{y'} &= \mathbf{A}\cdot\mathbf{j}' = (\mathbf{i}\cdot\mathbf{j}')A_x + (\mathbf{j}\cdot\mathbf{j}')A_y + (\mathbf{k}\cdot\mathbf{j}')A_z \\ A_{z'} &= \mathbf{A}\cdot\mathbf{k}' = (\mathbf{i}\cdot\mathbf{k}')A_x + (\mathbf{j}\cdot\mathbf{k}')A_y + (\mathbf{k}\cdot\mathbf{k}')A_z \end{aligned} \tag{1.8.3}$$

The scalar products $(\mathbf{i} \cdot \mathbf{i}')$, $(\mathbf{i} \cdot \mathbf{j}')$, and so on are called the *coefficients of transformation*.

The unprimed components are similarly expressed as

$$\begin{aligned} A_x &= \mathbf{A}\cdot\mathbf{i} = (\mathbf{i}'\cdot\mathbf{i})A_{x'} + (\mathbf{j}'\cdot\mathbf{i})A_{y'} + (\mathbf{k}'\cdot\mathbf{i})A_{z'} \\ A_y &= \mathbf{A}\cdot\mathbf{j} = (\mathbf{i}'\cdot\mathbf{j})A_{x'} + (\mathbf{j}'\cdot\mathbf{j})A_{y'} + (\mathbf{k}'\cdot\mathbf{j})A_{z'} \\ A_z &= \mathbf{A}\cdot\mathbf{k} = (\mathbf{i}'\cdot\mathbf{k})A_{x'} + (\mathbf{j}'\cdot\mathbf{k})A_{y'} + (\mathbf{k}'\cdot\mathbf{k})A_{z'} \end{aligned} \tag{1.8.4}$$

i. i' = **i'·i** and so on, but those in the rows (equations) of Equation 1.8.4 appear in the columns of terms in Equation 1.8.3, and conversely.

Thus, Equation 1.8.3 is written

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \mathbf{i}\cdot\mathbf{i}' & \mathbf{j}\cdot\mathbf{i}' & \mathbf{k}\cdot\mathbf{i}' \\ \mathbf{i}\cdot\mathbf{j}' & \mathbf{j}\cdot\mathbf{j}' & \mathbf{k}\cdot\mathbf{j}' \\ \mathbf{i}\cdot\mathbf{k}' & \mathbf{j}\cdot\mathbf{k}' & \mathbf{k}\cdot\mathbf{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{1.8.5}$$

The 3-by-3 matrix in Equation 1.8.5 is called the *transformation matrix*.

EXAMPLE 1.8.1

Express the vector $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ in terms of the triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$, where the $x'y'$–axes are rotated 45° around the z-axis, with the z- and z'-axes coinciding, as shown in Figure 1.8.1. Referring to the figure, we have for the coefficients of transformation $\mathbf{i} \cdot \mathbf{i}' = \cos 45°$ and so on; hence,

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{j}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{i}' = 0 \\ \mathbf{i}\cdot\mathbf{j}' = -1/\sqrt{2} & \mathbf{j}\cdot\mathbf{j}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{j}' = 0 \\ \mathbf{i}\cdot\mathbf{k}' = 0 & \mathbf{j}\cdot\mathbf{k}' = 0 & \mathbf{k}\cdot\mathbf{k}' = 1 \end{array}$$

These give

$$A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad A_{y'} = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad A_{z'} = 1$$

so that, in the primed system, the vector $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{5}{\sqrt{2}}\mathbf{i}' - \frac{1}{\sqrt{2}}\mathbf{j}' + \mathbf{k}'$$

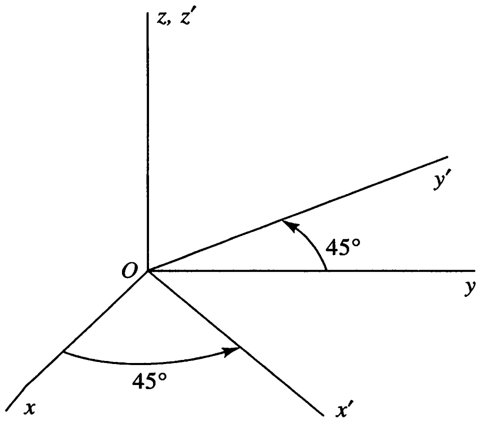


Figure 1.8.1 Rotated axes.

EXAMPLE 1.8.2

Find the transformation matrix for a rotation of the primed coordinate system through an angle ϕ about the z-axis. (Example 1.8.1 is a special case of this.) We have

$$\mathbf{i}\cdot\mathbf{i}' = \mathbf{j}\cdot\mathbf{j}' = \cos\phi$$
$$\mathbf{j}\cdot\mathbf{i}' = -\mathbf{i}\cdot\mathbf{j}' = \sin\phi$$
$$\mathbf{k}\cdot\mathbf{k}' = 1$$

and all other dot products are zero; hence, the transformation matrix is

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Consequently, the matrix for the combination of two rotations, the first being about the z-axis (angle ϕ) and the second being about the new y'-axis (angle θ), is given by the matrix product

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \tag{1.8.6}$$

Let us take the velocity v of a projectile of mass m traveling through space along a parabolic trajectory as an example of the vector.

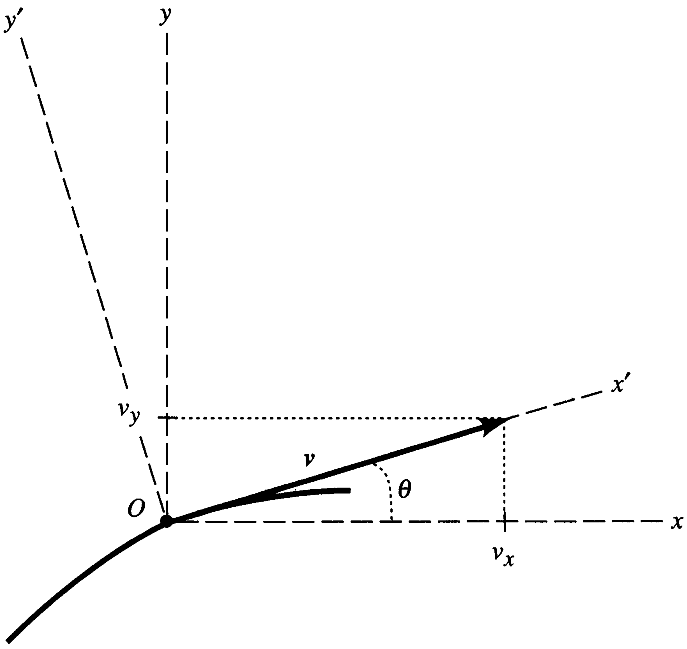


Figure 1.8.2 Velocity of a moving particle referred to two different two-dimensional coordinate systems.

We express the coordinate rotation in terms of the transformation matrix, defined in Equation 1.8.5. We write all vectors as column matrices; thus, the vector $\mathbf{v} = (v_x, v_y)$ is

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$$

Given the components in one coordinate system, we can calculate them in the other using the transformation matrix of Equation 1.8.5. We represent this matrix by the symbol **R**.[9]

$$\mathbf{R} = \begin{pmatrix} i\cdot i' & j\cdot i' \\ i\cdot j' & j\cdot j' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

The components of $\mathbf{v}'$ in the $x'y'$ coordinate system are

$$\mathbf{v}' = \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$$

or symbolically, $\mathbf{v}' = \mathbf{R}\mathbf{v}$. Here we have denoted the vector in the primed coordinate system by $\mathbf{v}'$. Bear in mind, though, that $\mathbf{v}$ and $\mathbf{v}'$ represent the same vector. The velocity vector

The square of the magnitude of $\mathbf{v}$ is

$$(\mathbf{v}\cdot\mathbf{v}) = \tilde{\mathbf{v}}\mathbf{v} = (v\cos\theta \quad v\sin\theta)\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix} = v^2\cos^2\theta + v^2\sin^2\theta = v^2$$

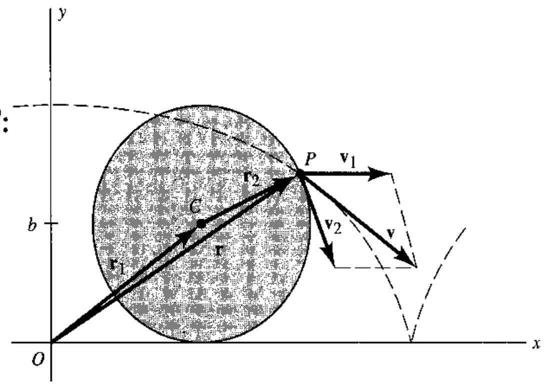
Similarly, the square of the magnitude of $\mathbf{v}'$ is

$$(\mathbf{v}'\cdot\mathbf{v}') = \tilde{\mathbf{v}}'\mathbf{v}' = (v \quad 0)\begin{pmatrix} v \\ 0 \end{pmatrix} = v^2 + 0^2 = v^2$$

In each case, the magnitude of the vector is a scalar v whose value is independent of our choice of coordinate system. The same is true of the mass of the projectile. If its mass is one kilogram in the xy coordinate system, then its mass is one kilogram in the $x'y'$ coor-

EXAMPLE 1.10.3

Rolling Wheel



Let us consider the following position vector of a particle P:

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$$

in which

$$\mathbf{r}_1 = \mathbf{i}b\omega t + \mathbf{j}b$$
$$\mathbf{r}_2 = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

Now $\mathbf{r}_1$ by itself represents a point moving along the line $y = b$ at constant velocity, provided ω is constant; namely,

$$\mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt} = \mathbf{i}b\omega$$

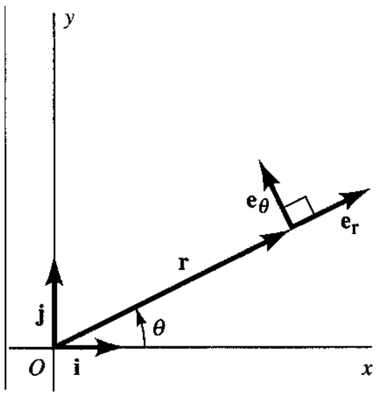
The second part, $\mathbf{r}_2$, is just the position vector for circular motion, as discussed in Example 1.10.2. Hence, the vector sum $\mathbf{r}_1 + \mathbf{r}_2$ represents a point that describes a circle of radius b about a moving center. This is precisely what occurs for a particle on the rim

the position vector of the particle P *relative* to the moving center. The actual path is a *cycloid*, as shown in Figure 1.10.5. The velocity of P is

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{i}(b\omega + b\omega \cos \omega t) - \mathbf{j}b\omega \sin \omega t$$

1.11 Velocity and Acceleration in Plane Polar Coordinates

polar coordinates r, θ to express the position of a particle moving in a plane. The position of the particle can be written as the product of the radial distance r by a unit radial vector e_r :



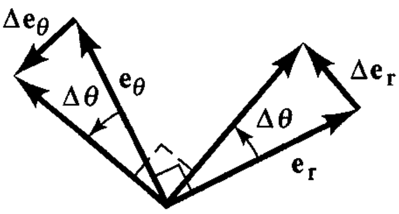
$$\mathbf{r} = r\mathbf{e}_r$$

if we differentiate with respect to t, we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

To calculate the derivative $d\mathbf{e}_r/dt$,

Let us introduce another unit vector, $\mathbf{e}_\theta$, whose direction is perpendicular to $\mathbf{e}_r$.



$$\Delta\mathbf{e}_r \simeq \mathbf{e}_\theta \Delta\theta$$

If we divide by Δt and take the limit, we get

$$\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta \frac{d\theta}{dt}$$

$$\Delta\mathbf{e}_\theta \simeq -\mathbf{e}_r \Delta\theta$$

$$\frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r \frac{d\theta}{dt}$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

Thus, $\dot{r}$ is the radial component of the velocity vector, and $r\dot{\theta}$ is the transverse component.

To find the acceleration vector, we take the derivative of the velocity with respect to time. This gives

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{e}_r + \dot{r}\frac{d\mathbf{e}_r}{dt} + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt}$$

The values of $d\mathbf{e}_r/dt$ and $d\mathbf{e}_\theta/dt$ are given the acceleration vector in plane polar coordinates:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

Thus, the radial component of the acceleration vector is

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and the transverse component is

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

EXAMPLE 1.11.1

A honeybee hones in on its hive in a spiral path in such a way that the radial distance decreases at a constant rate, $r = b - ct$, while the angular speed increases at a constant rate, $\dot{\theta} = kt$. Find the speed as a function of time.

Solution:

We have $\dot{r} = -c$ and $\ddot{r} = 0$. Thus, from Equation 1.11.7,

$$\mathbf{v} = -c\mathbf{e}_r + (b - ct)kt\mathbf{e}_\theta$$

so

$$v = [c^2 + (b - ct)^2k^2t^2]^{1/2}$$

which is valid for $t \leq b/c$. Note that $v = c$ both for $t = 0$, $r = b$ and for $t = b/c$, $r = 0$.

EXAMPLE 1.11.2

On a horizontal turntable that is rotating at constant angular speed, a bug is crawling outward on a radial line such that its distance from the center increases quadratically with time: $r = bt^2$, $\theta = \omega t$, where b and ω are constants. Find the acceleration of the bug.

Solution:

We have $\dot{r} = 2bt$, $\ddot{r} = 2b$, $\dot{\theta} = \omega$, $\ddot{\theta} = 0$. Substituting into Equation 1.11.9, we find

$$\begin{aligned}\mathbf{a} &= \mathbf{e}_r(2b - bt^2\omega^2) + \mathbf{e}_\theta[0 + 2(2bt)\omega] \\ &= b(2 - t^2\omega^2)\mathbf{e}_r + 4b\omega t\mathbf{e}_\theta\end{aligned}$$

Note that the radial component of the acceleration becomes negative for large t in this example, although the radius is always increasing monotonically with time.

1.12 Velocity and Acceleration in Cylindrical and Spherical Coordinates

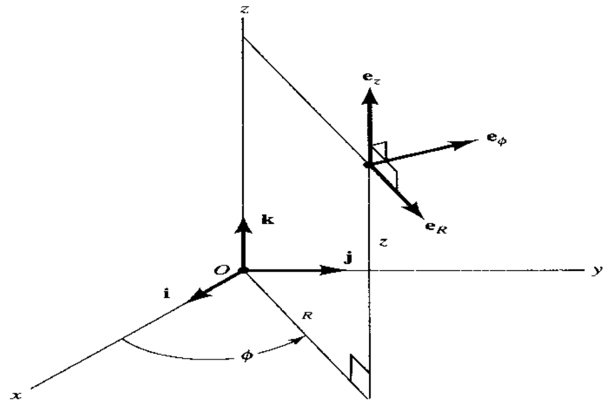
Cylindrical Coordinates

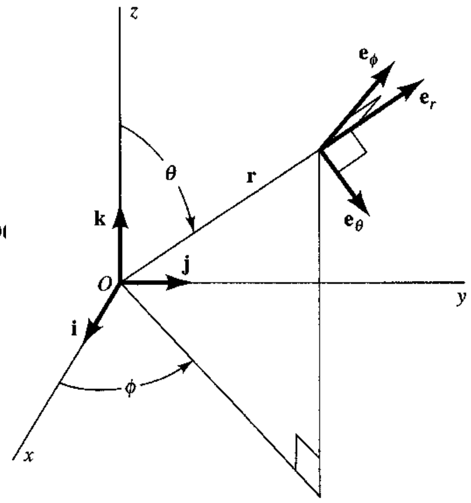
cylindrical coordinates R, ϕ, z. The position vector is then written as

$$\mathbf{r} = R\mathbf{e}_R + z\mathbf{e}_z$$

$\mathbf{v} = \dot{R}$

$\mathbf{a} = ($





Spherical Coordinates

When spherical coordinates r, θ, ϕ are employed to describ

$$\mathbf{r} = r\mathbf{e}_r$$

$$\mathbf{v} = \mathbf{e}_r\dot{r} + \mathbf{e}_\phi r\dot{\phi}\sin\theta + \mathbf{e}_\theta r\dot{\theta}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta$$
$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\mathbf{e}_\phi$$

EXAMPLE 1.12.1

A bead slides on a wire bent into the form of a helix, the motion of the bead being given in cylindrical coordinates by $R = b$, $\phi = \omega t$, $z = ct$. Find the velocity and acceleration vectors as functions of time.

Solution:

Differentiating, we find $\dot{R} = \ddot{R} = 0$, $\dot{\phi} = \omega$, $\ddot{\phi} = 0$, $\dot{z} = c$, $\ddot{z} = 0$. So, from Equations 1.12.2 and 1.12.3, we have

$$\mathbf{v} = b\omega\mathbf{e}_\phi + c\mathbf{e}_z$$
$$\mathbf{a} = -b\omega^2\mathbf{e}_R$$

b

1.17 A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b\cos\omega t + \mathbf{j}2b\sin\omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and at $t = \pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

1.18 A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b\sin\omega t + \mathbf{j}b\cos\omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

1.19 A bee goes out from its hive in a spiral path given in plane polar coordinates by

$$r = be^{kt} \qquad \theta = ct$$

where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (*Hint: Find* $\mathbf{v}\cdot\mathbf{a}/va$.)

1.20 Work Problem 1.18 using cylindrical coordinates where $R = b$, $\phi = \omega t$, and $z = ct^2$.

1.21 The position of a particle as a function of time is given by

$$\mathbf{r}(t) = \mathbf{i}(1 - e^{-kt}) + \mathbf{j}e^{kt}$$

where k is a positive constant. Find the velocity and acceleration of the particle. Sketch its trajectory.

1.17 $\vec{v}(t) = -\hat{i}b\omega\sin(\omega t) + \hat{j}2b\omega\cos(\omega t)$

$$|\vec{v}| = \left(b^2\omega^2\sin^2\omega t + 4b^2\omega^2\cos^2\omega t\right)^{\frac{1}{2}} = b\omega\left(1+3\cos^2\omega t\right)^{\frac{1}{2}}$$

$$\vec{a}(t) = -\hat{i}b\omega^2\cos\omega t - \hat{j}2b\omega^2\sin\omega t$$

$$|\vec{a}| = b\omega^2\left(1+3\sin^2\omega t\right)^{\frac{1}{2}}$$

at $t=0$, $|\vec{v}| = 2b\omega$; at $t = \dfrac{\pi}{2\omega}$, $|\vec{v}| = b\omega$

1.18 $\vec{v}(t) = \hat{i}b\omega\cos\omega t - \hat{j}b\omega\sin\omega t + \hat{k}2ct$

$$\vec{a}(t) = -\hat{i}b\omega^2\sin\omega t - \hat{j}b\omega^2\cos\omega t + \hat{k}2c$$

$$|\vec{a}| = \left(b^2\omega^4\sin^2\omega t + b^2\omega^4\cos^2\omega t + 4c^2\right)^{\frac{1}{2}} = \left(b^2\omega^4 + 4c^2\right)^{\frac{1}{2}}$$

1.19 $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = bke^{kt}\hat{e}_r + bce^{kt}\hat{e}_\theta$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta = b\left(k^2-c^2\right)e^{kt}\hat{e}_r + 2bcke^{kt}\hat{e}_\theta$$

$$\cos\phi = \frac{\vec{v}\cdot\vec{a}}{va} = \frac{b^2k\left(k^2-c^2\right)e^{2kt} + 2b^2c^2ke^{2kt}}{be^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}be^{kt}\left[\left(k^2-c^2\right)^2 + 4c^2k^2\right]^{\frac{1}{2}}}$$

$$\cos\phi = \frac{k\left(k^2+c^2\right)}{\left(k^2+c^2\right)^{\frac{1}{2}}\left(k^2+c^2\right)} = \frac{k}{\left(k^2+c^2\right)^{\frac{1}{2}}}, \quad \text{a constant}$$

1.20 $\vec{a} = \left(\ddot{R} - R\dot{\phi}\right)\hat{e}_R + \left(2\dot{R}\dot{\phi} + R\ddot{\phi}\right)\hat{e}_\phi + \ddot{z}\hat{e}_z$

$$\vec{a} = -b\omega^2\hat{e}_R + 2c\hat{e}_z$$

$$|\vec{a}| = \left(b^2\omega^4 + 4c^2\right)^{\frac{1}{2}}$$

1.21 $\vec{r}(t) = \hat{i}\left(1-e^{-kt}\right) + \hat{j}e^{kt}$

$$\dot{\vec{r}}(t) = \hat{i}ke^{-kt} + \hat{j}ke^{kt}$$

$$\ddot{\vec{r}}(t) = -\hat{i}k^2e^{-kt} + \hat{j}k^2e^{kt}$$