Vectors & Scalars

Vectors

- A scalar can be completely defined by *magnitude*.
- Examples: mass, density, volume, and energy.
- Its value is independent of any chosen coordinates.
- Mathematically, scalars obey the normal algebraic rules of addition, multiplication ...etc.

- A vector needs both *magnitude* and *direction* to be defined.
- Examples: displacement, velocity, acceleration, and force.
- Its value is coordinate system dependent.
- Mathematically, vectors need a special treatment known as ***Vectors Algebra***.

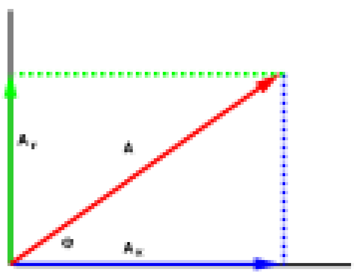
Basic method to describe vector is Cartesian coordinate system.

Vector Algebra

A given vector **A** can be specified by:

Its ***magnitude*** (A) and its ***direction*** (ϕ) relative to some chosen coordinate system.

The set of its ***components***, or projections onto the coordinate axes ; Since



and

$$A = |A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$ ***The magnitude***

$$\phi = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$ ***The direction***

Vector Addition:

The addition of two vectors is defined by the equation:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x,\ A_y + B_y,\ A_z + B_z)$$

Note that:

A+B = B+A Commutative Law

A+(B+C) = (A+B)+C Associative Law

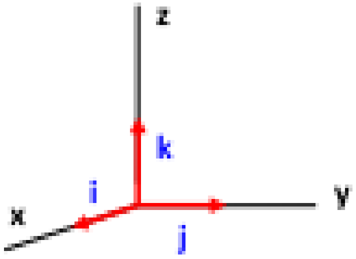
The Unit Vectors:

A unit vector is a vector whose magnitude is **unity**. Unit vectors are often assigned by the symbol **e**, The three unit vectors

$\mathbf{e}_x = (1,0,0)$ $\mathbf{e}_y = (0,1,0)$ $\mathbf{e}_z = (0,0,1)$

for **Cartesian coordinates**

$\mathbf{e}_x = \mathbf{i}$ $\mathbf{e}_y = \mathbf{j}$ $\mathbf{e}_z = \mathbf{k}$



The Scalar Product:

Given two vectors **A** and **B**, the ***scalar or " dot " product***, **A.B** is the scalar defined by the equation

$$\mathbf{A.B} = A_x B_x + A_y B_y + A_z B_z$$

Main properties of the scalar product:

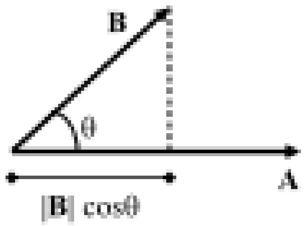
- It is **commutative**, **A.B = B.A**
- It is **distributive**, **A . (B + C) = A . B + A . C**
- It is a **scalar**. If vector **A** is expressed as $(A_x, 0, 0)$ and the vector **B** as $(B_x, B_y, 0)$ or $(B \cos\theta, B \sin\theta, 0)$, then,

$$\mathbf{A.B} = A_x B_x = A(B \cos\theta) = |\mathbf{A}|\,|\mathbf{B}| \cos\theta$$

- Hence, The geometrical interpretation of **A . B** is that it is the projection of **B** onto **A** times the length of **A**.
- If **A. B** = 0 , and neither **A** nor **B** is null, then ($\cos\theta = 0$) and **A** is ⊥ to **B**.

Similarly;

$$\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1$$
$$\mathbf{i.j} = \mathbf{j.k} = \mathbf{k.i} = 0$$



The Vector Product

Given two vectors **A** and **B**, the ***vector or "cross " product*** , **A** x **B** is a **vector** whose components are given by the equation

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y\,,\ \ A_z B_x - A_x B_z\,,\ \ A_x B_y - A_y B_x\,)$$

which is equal to the *determinant*,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \vec{j}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \vec{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \vec{i}\,(A_y B_z - A_z B_y) + \vec{j}\,(A_z B_x - A_x B_z) + \vec{k}\,(A_x B_y - A_y B_x)$$

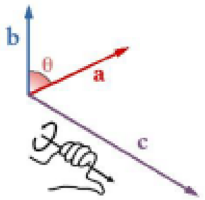
Main properties of the *vector product*:

- It is ***anti-commutative***, $\mathbf{A} \times \mathbf{B} = -\,\mathbf{B} \times \mathbf{A}$
- It is ***distributive***, $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
- the resultant is a ***vector*** . Its ***magnitude*** is given by;

$$|\,\mathbf{A} \times \mathbf{B}\,| = |A|\ |B|\ \sin\theta$$

where θ is the **smallest** angle between **A** and **B**.

The Vector Product



- The ***direction*** of the resultant vector is ⊥ to the plane containing **A** and **B**.

Hence,

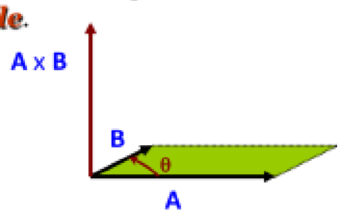
$$\mathbf{A} \times \mathbf{B} = (A\,B\,\sin\theta)\,\mathbf{n}$$

where **n** is a unit vector normal to the plane containing **A** and **B**. The sense of **n** is given by the ***right –hand rule***.

Therefore,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

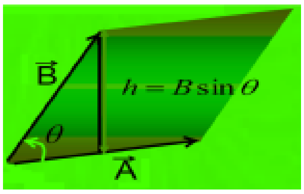
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$



The *cross product* $\mathbf{A} \times \mathbf{B}$ has

1- A magnitude of $A\,B\,\sin\theta$ which is equal to the area of the parallelogram with sides **A** and **B** shown by the shaded area in the Figure.

2- A direction ⊥ to the plane containing **A** and **B**.



Derivative of a vector

Previously we were concerned mainly with vector algebra. Now, we will begin to study the calculus of vectors and its use in the description of the motion of particles.

Consider a vector **A**, whose components are functions of a single variable u which is usually the time t, i.e,

$$\mathbf{A}(u) = \mathbf{i}\,A_x(u) + \mathbf{j}\,A_y(u) + \mathbf{k}\,A_z(u)$$

The derivative of **A** with respect to u is defined by

$$\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}$$

The rules for differentiating vector products

$$\frac{d}{du}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du} \qquad \frac{d}{du}(\mathbf{A}.\mathbf{B}) = \frac{d\mathbf{A}}{du}.\mathbf{B} + \mathbf{A}.\frac{d\mathbf{B}}{du}$$

$$\frac{d}{du}(n\mathbf{A}) = \frac{dn}{du}\mathbf{A} + n\frac{d\mathbf{A}}{du} \qquad \frac{d}{du}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{du} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{du}$$

Notice that it is necessary to maintain the order of the terms in the derivative of the cross product.

The position vector

Velocity & Acceleration In rectangular coordinates

In a given reference system, ***the position of a particle*** can be specified by a single vector. This vector is called *the position vector* of the particle.

In rectangular coordinates (cartesian coordinates), the *position vector* is simply

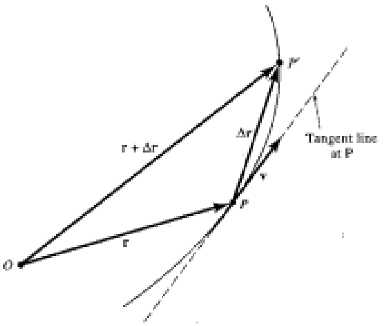
$$\mathbf{r} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$$

The velocity vector

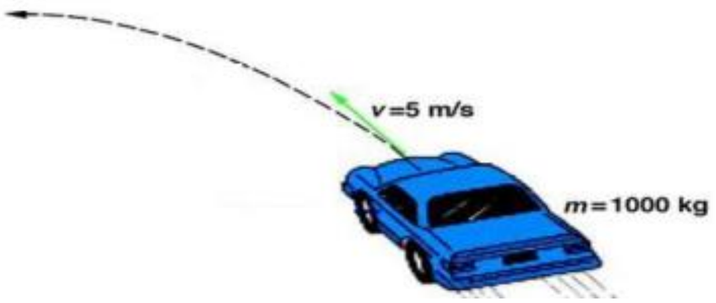
For a moving particle, these components are functions of the time.
The time derivative of r *is* called the ***velocity*, (v)**, which is given by:
($\dot{x}$=dx/dt, $\dot{y}$=dy/dt and $\dot{z}$=dz/dt).

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\,\dot{x} + \mathbf{j}\,\dot{y} + \mathbf{k}\,\dot{z}$$

The vector d**r**/dt expresses both the *direction* and the *rate* of motion. As Δt approaches zero, the point P' approaches P, and the direction of the vector Δ**r**/Δt approaches the direction of the tangent to the path at P, which is d**r**/dt.



Note: The velocity vector is always tangent to the path of motion.



The **magnitude** of the velocity is called the *speed* (v). In rectangular components the speed is just

$$v = \frac{ds}{dt} = |\mathbf{v}| = \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)^{1/2}$$

where s is the distance.

The acceleration vector

The time derivative of the velocity is called the ***acceleration*** (**a**). Hence;

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{i}\,\ddot{x} + \mathbf{j}\,\ddot{y} + \mathbf{k}\,\ddot{z}$$

EXAMPLE 1.10.1

Projectile Motion

Let us examine the motion represented by the equation

$$\mathbf{r}(t) = \mathbf{i}bt + \mathbf{j}\left(ct - \frac{gt^2}{2}\right) + \mathbf{k}0$$

This represents motion in the xy plane, because the z component is constant and equal to zero. The velocity $\mathbf{v}$ is obtained by differentiating with respect to t, namely,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b + \mathbf{j}(c - gt)$$

The acceleration, likewise, is given by

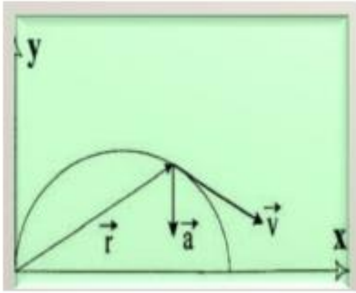
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{j}g$$

Thus, $\mathbf{a}$ is in the negative y direction and has the constant magnitude g. The path of motion is a parabola, as shown in Figure 1.10.3. The speed v varies with t according to the equation

$$v = [b^2 + (c - gt)^2]^{1/2}$$

Exp(1-10-1):**Projectile Motion**

Examples of Velocity & Acceleration In rectangular coordinates



The position vector	$\vec{r} = i\,bt + j\,(ct - \frac{gt^2}{2})$
The velocity	$\vec{v} = i\,b + j\,(c - gt)$
The acceleration	$\vec{a} = -j\,g$
The path	**Parabola**

EXAMPLE 1.10.2

Circular Motion

Suppose the position vector of a particle is given by

$$\mathbf{r} = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

where ω is a constant.

Let us analyze the motion. The distance from the origin remains constant:

$$|\mathbf{r}| = r = (b^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{1/2} = b$$

So the path is a circle of radius b centered at the origin. Differentiating $\mathbf{r}$, we find the velocity vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b\omega \cos \omega t - \mathbf{j}b\omega \sin \omega t$$

The particle traverses its path with constant speed:

$$v = |\mathbf{v}| = (b^2\omega^2 \cos^2 \omega t + b^2\omega^2 \sin^2 \omega t)^{1/2} = b\omega$$

The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{i}b\omega^2 \sin \omega t - \mathbf{j}b\omega^2 \cos \omega t$$

In this case the acceleration is perpendicular to the velocity, because the dot product of $\mathbf{v}$ and $\mathbf{a}$ vanishes:

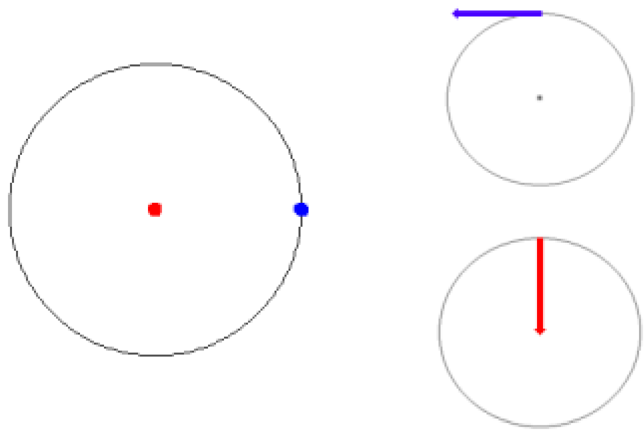
$$\mathbf{v} \cdot \mathbf{a} = (b\omega \cos \omega t)(-b\omega^2 \sin \omega t) + (-b\omega \sin \omega t)(-b\omega^2 \cos \omega t) = 0$$

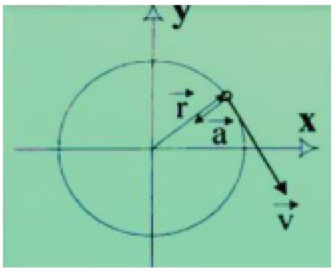
Comparing the two expressions for $\mathbf{a}$ and $\mathbf{r}$, we see that we can write

$$\mathbf{a} = -\omega^2 \mathbf{r}$$

Exp(1-10-2):

Circular Motion





The position vector	$\mathbf{r} = \mathbf{i}\, b\cos\omega t + \mathbf{j}\, b\sin\omega t$
The velocity	$\mathbf{v} = -\mathbf{i}\, b\omega\sin\omega t + \mathbf{j}\, b\omega\cos\omega t$
The acceleration	$\mathbf{a} = -\mathbf{i}\, b\omega^2\cos\omega t - \mathbf{j}\, b\omega^2\sin\omega t$ $\mathbf{a} = -\omega^2\,\mathbf{r}$ $\quad v \perp a$
The path	**Circle**

H.W. Please make sure that **v.a**=0