Example: The map-germs $y = x^2$ and $y = x^4$ at the point 0 of the real line are topologically equivalent. The germ $y = x^2$ is topologically (and even differentiably) stable at zero. The germ $y = x^4$ is differentiably (and even topologically) unstable at zero.

Two maps $f, g : \mathbb{R}^n \rightarrowtail \mathbb{R}^p$ are said to be ***germ-equivalent*** at $p \in \mathbb{R}^n$ if p is in the domain of both and there is a neighbourhood U of p such that the restrictions coincide: $f|_U = g|_U$; that is, if,

$$\forall x \in U,\ f(x) = g(x).$$

A ***map-germ*** or ***function-germ*** at a point p is an equivalence class of germ-equivalent maps. If η is such an equivalence class, then any $f \in \eta$ is called a ***representative*** of η.

Notation Given a map $f : \mathbb{R}^n \rightarrowtail \mathbb{R}^p$ with $x \in \operatorname{dom} f$, we denote the germ of f at x by $[f]_x$. In other words, $[f]_x$ is the set of all maps $g : \mathbb{R}^n \rightarrowtail \mathbb{R}^p$ with $x \in \operatorname{dom} g$ and for which there is a neighbourhood U of x such that $f|_U = g|_U$.

Definition We define a ***singularity*** of a smooth map $f : \mathbb{R}^n \rightarrowtail \mathbb{R}^p$ to be a point $x_0 \in \mathrm{dom}(f)$ where the rank is not maximal (ie, not equal to $\min\{n,p\}$). If $p = 1$ (so scalar-valued functions), this means a point where all partial derivatives vanish[2].

A first step in the study of singularities is to introduce coordinates adapted to the situation.

Theorem *Let $[f]_0 : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth map-germ at the origin, with $\mathrm{rk}_0(f) = k$. Then there are coordinates $u_1, \dots, u_k, x_1, \dots, x_{n-k}$ on $\mathbb{R}^n$ and $(y_1, \dots, y_p)$ on $\mathbb{R}^p$, and a smooth map-germ $[g]_0 : (\mathbb{R}^n, 0) \to (\mathbb{R}^{p-k}, 0)$ with $\mathrm{d}g_0 = 0$ and $g(u, 0) = 0$, such that $[f]_0$ takes the form,*

$$f(u,x) = (u, g(u,x)).$$

That is, in a neighbourhood of 0, a representative map f is given by

$$\begin{cases} y_i = u_i, & (i = 1, \dots, k) \\ y_j = g_{j-k}(x,u) & (j = k+1, \dots, p) \end{cases}$$

The ring of germs of smooth functions

Denote by $\mathcal{E}_n$ the set of all germs at the origin of smooth functions on $\mathbb{R}^n$. It has a natural ring structure, given by addition and multiplication of functions as follows. Let f and g be smooth functions with 0 in their domains, and let U be any neighbourhood of 0 with $U \subset \operatorname{dom} f \cap \operatorname{dom} g$ (so that f and g are both defined on U) then the operations are

$$[f]_0 + [g]_0 := \left[f_{|U} + g_{|U}\right]_0, \quad \text{and} \quad [f]_0[g]_0 := \left[f_{|U} g_{|U}\right]_0.$$

The ring $\mathcal{E}_n$ has an important ideal $\mathfrak{m}_n$ consisting of all smooth function germs vanishing[1] at the origin:

$$\mathfrak{m}_n = \{f \in \mathcal{E}_n \mid f(0) = 0\}.$$

given any collection $f_1, \dots, f_r$ of elements of $\mathcal{E}_n$, then the ideal generated by $f_1, \dots, f_r$, which we write as $\langle f_1, f_2, \dots, f_r \rangle$, is the set of all "linear combinations" of these generators, with coefficients taken from the ring. That is,

$$\langle f_1, f_2, \ldots, f_r\rangle = \left\{\sum_{j=1}^{r} a_j f_j \mid a_j \in \mathcal{E}_n\right\}.$$

Examples Let J be the ideal $J = \langle x^2 + x^3\rangle \subset \mathcal{E}_1$, and $I = \langle x^2\rangle$.

A short calculation shows that $I = J$. Indeed, $x^2+x^3 = (1+x)x^2 \in I$ so that $J \subset I$, and moreover $(1+x)$ is invertible in $\mathcal{E}_1$ (because it is not in $\mathfrak{m}_1$) and so $x^2 = (1+x)^{-1}(x^2+x^3) \in J$, so $I \subset J$. So we have $J = I$.

Proposition Let $\mathfrak{m}_n = \langle x_1, x_2, \ldots, x_n\rangle$. and

Write $\mathbb{R}^n = \mathbb{R}^a \times \mathbb{R}^b$ (with $a + b = n$), and use coordinates $(x_1, \ldots, x_a, y_1, \ldots, y_b)$. The ideal $I = \{f(x,y) \in \mathcal{E}_n \mid f(0,y) \equiv 0\}$ of functions vanshing on $\mathbb{R}^b$ satisfies

$$I = \langle x_1, x_2, \ldots, x_a\rangle.$$

Hadamard's lemma is obtained from this by putting $b = 0$, in which case $a = n$ and $I = \mathfrak{m}_n$.

Powers of the maximal ideal are defined inductively: for $r \geq 2$,

$$\mathfrak{m}_n^r := \mathfrak{m}_n.\mathfrak{m}_n^{r-1}.$$

Corollary 3.6 *The germ $f \in \mathcal{E}_n$ is in $\mathfrak{m}_n^r$ if and only if f and all of its partial derivatives of order less than r vanish at the origin.*

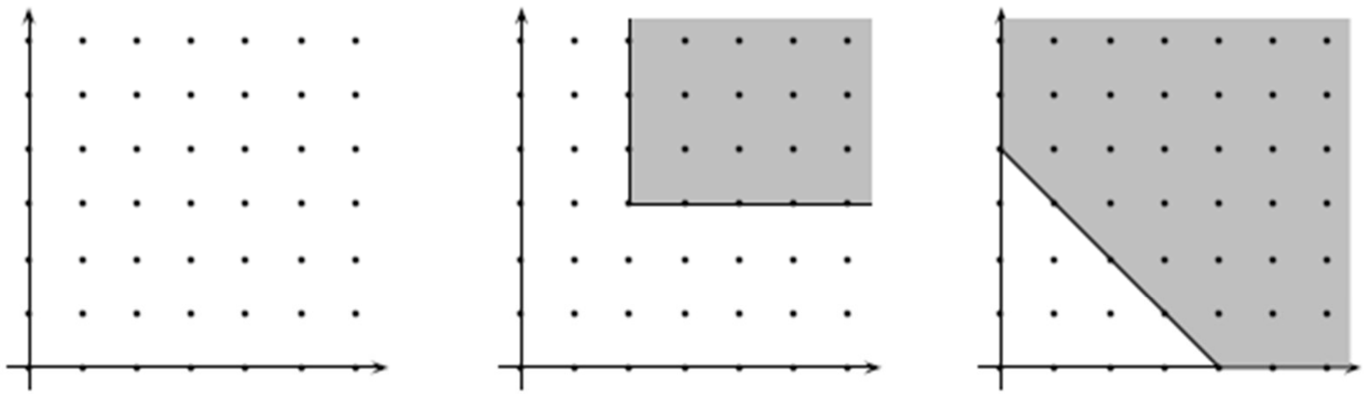
Rings and jets The k-jet of a function germ is its kth order Taylor series. That means that starting with the infinite Taylor series, we ignore all terms of degree greater than k It follows that one can identify the k-jet of a germ $f \in \mathcal{E}_n$ with the image of f under the projection π_k, where

$$\pi_k : \mathcal{E}_n \to \mathcal{E}_n/\mathfrak{m}_n^{k+1}$$

is the ring homomorphism, with kernel $\mathfrak{m}_n^{k+1}$.

Newton diagram

The skeleton of the Newton diagram for $\mathcal{E}_2$ is the lattice $\mathbb{N} \times \mathbb{N}$ (where $\mathbb{N}$ includes 0) of points in $\mathbb{R}^2$ with non-negative integer coefficients. Each point of the lattice represents a monomial, with (a, b) representing $x^a y^b \in \mathcal{E}_2$. See Figure 3.1(a). More generally, for $\mathcal{E}_n$ it is the lattice $\mathbb{N}^n$, with $(a_1, a_2, \ldots, a_n)$ representing the monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.



(a) The skeleton of the Newton diagram for $\mathcal{E}_2$

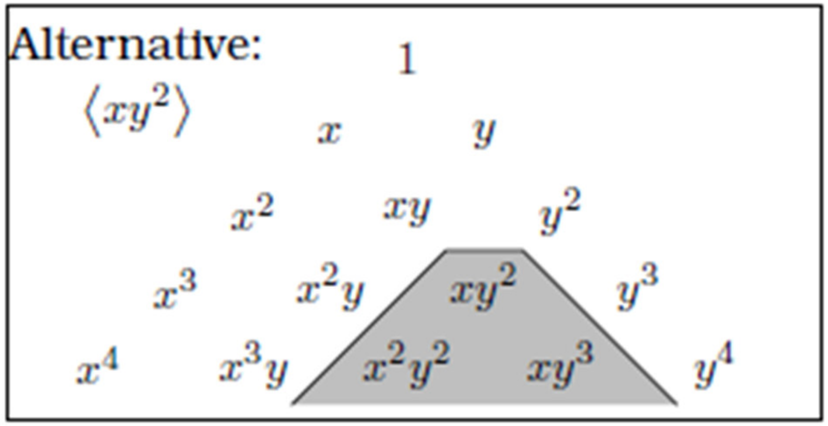
(b) Newton diagram for the ideal $\langle x^2y^3 \rangle$

(c) Newton diagram for the ideal $\mathfrak{m}_2^4$

Figure 3.1: Newton diagrams for a few ideals

Suppose now that I is an ideal generated by a single monomial $I = \langle x^a y^b \rangle$. Then every monomial $x^c y^d \in I$ if and only if $c \geq a$ and $d \geq b$, so I contains all the monomials corresponding to (c, d) above and to the right of (a, b). So we shade that region of the lattice. See Figure 3.1(b) for the ideal $\langle x^2y^3 \rangle$. Figure 3.1(c) shows the Newton diagram for the ideal $\mathfrak{m}_2^4$.

An alternative way of depicting Newton diagrams is shown below.



Ideals of finite codimension

Definition 3.8 An ideal I is of ***finite codimension*** if $\mathcal{E}_n/I$ is a finite dimensional vector space. ✓

This means that there is a finite dimensional vector subspace V of $\mathcal{E}_n$ such that $\mathcal{E}_n = V + I$, so any germ $f \in \mathcal{E}_n$ can be written as $f = g + h$ with $g \in V$ and $h \in I$.

Example 3.9 The maximal ideal $\mathfrak{m}_n$ is of finite codimension in $\mathcal{E}_n$ because

$$\mathcal{E}_n = \mathfrak{m}_n + \mathbb{R}.$$

So $V = \mathbb{R}$. This is because for any $f \in \mathcal{E}_n$ one has $f(0) \in \mathbb{R}$ and we let $\bar{f}$ be the germ $\bar{f} = f - f(0)$ which is in $\mathfrak{m}_n$. Then f can be written as

$$f = \bar{f} + f(0).$$

Proposition 3.10 *An ideal I is of finite codimension if and only if there is $r \in \mathbb{N}$ such that $\mathfrak{m}_n^r \subset I$.*

Lemma 3.11 (Nakayama) *Let I, J be ideals in $\mathcal{E}_n$ with I finitely generated. Then*

$$I \subset J + \mathfrak{m}_n I \Rightarrow I \subset J.$$

Example 3.12 We show $\mathfrak{m}_2^5 \subset \langle x^3, y^3 + x^2y^2 \rangle$. Let $I = \mathfrak{m}_2^5$ and $J = \langle x^3, y^3 + x^2y^2 \rangle$. We want to show that $I \subset J + \mathfrak{m}_2 I$, which is $J + \mathfrak{m}_2^6$, and then apply Nakayama's lemma. Check each generator of I in turn:

$$x^5 = x^2(x^3), \quad x^4y = xy(x^3), \quad x^3y^2 = y^2(x^3),$$
$$x^2y^3 = x^2(y^3 + x^2y^2) - x^4y^2, \quad xy^4 = xy(y^3 + x^2y^2) - x^3y^3,$$
$$y^5 = y^2(y^3 + x^2y^2) - x^2y^4.$$

The first three show directly that the monomials belong to J, while the others are all written in the form $p = q(y^3 + x^2y^2) - r$ with $r \in \mathfrak{m}_2^6$, so showing that $p \in J + \mathfrak{m}^6$. By Nakayama's lemma we are done.

Codimension

Definition 4.3 A germ in $\mathcal{E}_n$ is said to be of ***finite codimension*** if the Jacobian ideal is of finite codimension in $\mathcal{E}_n$. If f has a critical point at the origin then $f \in \mathfrak{m}_n^2$ and in this case $Jf \subset \mathfrak{m}_n$ and one says its ***codimension*** is

$$\operatorname{cod}(f) := \dim \left(\frac{\mathfrak{m}_n}{Jf} \right).$$

This number is finite if and only if f is of finite codimension. ✓

Examples 4.4 (i). If $f = x_1^2 + x_2^2 + \cdots + x_n^2 \in \mathcal{E}_n$ then $Jf = \mathfrak{m}_n$, so $\mathrm{cod}(f) = 0$.

(ii). For $f = x^3 + y^3 \in \mathcal{E}_2$ one has $Jf = \langle x^2, y^2 \rangle$; a basis for $\mathfrak{m}_2/Jf$ can be taken to be $\{x, y, xy\}$ so f has codimension 3.

(iii). Let $f(x, y) = x^2 y$. Then $Jf = \langle xy, x^2 \rangle$ and f is of infinite codimension, as for all $k > 0$, $y^k \notin Jf$.