Singularity theory

The simplest examples

Example: The critical point 0 of the function $y = x^2$ is nondegenerate, while the critical point 0 of the function $y = x^3$ is degenerate (Fig. 1).

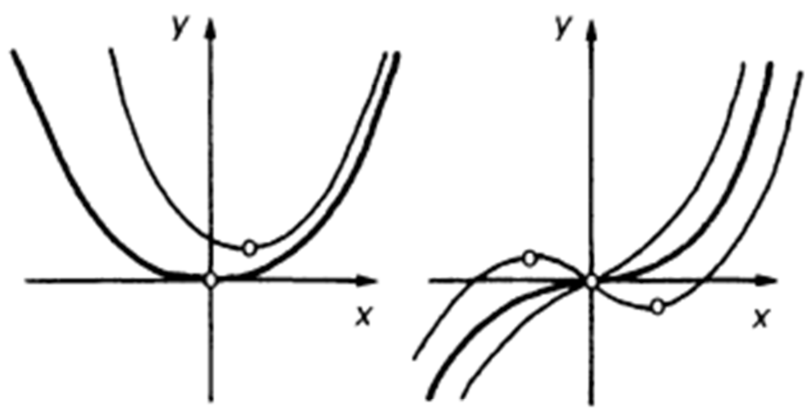


Fig. 1.

Consider an arbitrary smooth function, close (along with its derivatives) to the function $y = x^2$. It is clear that near zero this function will have a critical point, similar to the critical point of $y = x^2$. The critical point of $y = x^2$ is stable in the sense that under small perturbations of the function it does not vanish, but simply shifts slightly.

The degenerate critical point of the function $y = x^3$ behaves completely differently under small perturbations.

Example: Consider the family of functions of one variable $y = x^3 + \varepsilon x$. For small ε the functions of this family can be considered as small perturbations of $y = x^3$. We see that under this perturbation the degenerate critical point either vanishes (for $\varepsilon > 0$) or decomposes into two nondegenerate critical points at a distance of order $\sqrt{|\varepsilon|}$ from it (for $\varepsilon < 0$).

Critical points and critical values of smooth maps

Consider a differentiable map $f: M^m \to N^n$. First of all we must extend to this case the concept of critical point. The derivative of a map f at a point x is a linear map of the tangent space of the source manifold at the point x to the tangent space of the target manifold at the point $f(x)$:

$$f_{*x}: T_x M^m \to T_{f(x)} N^n.$$

Definition: A point x of the manifold M is said to be a *critical point* of the smooth map $f: M \to N$ if the rank of the derivative

$$f_{*x}: T_x M \to T_{f(x)} N$$

at that point is less than the maximum possible value, that is less than the smaller of the dimensions of M and N:

$$\operatorname{rank} f_{*x} < \min(\dim M, \dim N).$$

Example: For the projection of the sphere to the horizontal plane the critical points are the points of the horizontal equator. Off the equator the rank of the derivative is equal to 2, while at points x of the equator the rank of the operator f_{*x} falls to 1.

Differentiable equivalence

If $f_p: M_p \to N_p$, $p = 1, 2$ are two given maps then to say that they are topologically equivalent means that there exist homeomorphisms $h: M_1 \to M_2$ and $k: N_1 \to N_2$ such that $f_2 = k f_1 h^{-1}$.

In other words a topological equivalence is a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & N_1 \\ h\downarrow & & \downarrow k \\ M_2 & \xrightarrow{f_2} & N_2 \end{array}$$

Definition: A *differentiable equivalence* of differentiable maps $f_1 : M_1 \to N_1$ and $f_2 : M_2 \to N_2$ is a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & N_1 \\ h\downarrow & & \downarrow k \\ M_2 & \xrightarrow{f_2} & N_2 \end{array}$$

whose vertical arrows are diffeomorphisms (differentiable one-to-one maps whose inverses also are differentiable*).

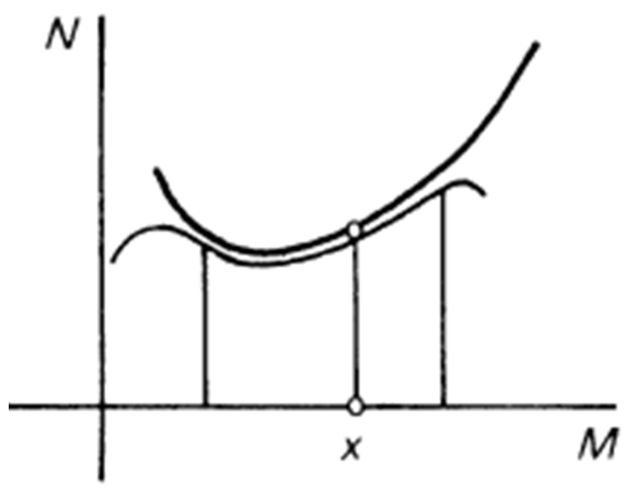
Stability

Consider a smooth map $f: M \to N$ of a closed manifold M to a manifold N.

Definition: A map f is said to be *differentiably stable* (or more precisely *left-right-differentiably stable*, or briefly simply *stable*), if every map sufficiently close to it† is differentiably equivalent to it.

Definition: A *map-germ* $M \to N$ at a point x of M is an equivalence class of maps $\varphi: U \to N$ (each of which is defined on some neighbourhood U of x in M, not necessarily the same for each); here two maps are regarded as equivalent if they coincide on some neighbourhood of the point x

Two maps of the same class are said to have the *same germ at the point* x



Definition: Two smooth map-germs are said to be (*left-right, differentiably*) *equivalent* if there are germs of diffeomorphisms of the source and target spaces transforming the first germ into the second (if the map-germ f_1 at x_1 is equivalent to the map-germ f_2 at x_2 then there exist a diffeomorphism-germ h at x_1 sending x_1 to x_2 and a diffeomorphism-germ k at $f_1(x_1)$ sending $f_1(x_1)$ to $f_2(x_2)$, such that $k(f_1(h^{-1}(x))) \equiv f_2(x)$ in a sufficiently small neighbourhood of x_2). The equivalence class of a germ at a critical point is said to be a *singularity*.

Definition: A smooth map-germ $f: M \to N$ at a point x of M (Fig. 6) is said to be (*left-right, differentiably*) *stable* if for a sufficiently small neighbourhood U of x there is a neighbourhood E of the map* f in $\Omega(M, N)$, such that for any map $\tilde{f}$ in E there is a point $\tilde{x}$ in U such that the germ of $\tilde{f}$ at $\tilde{x}$ is equivalent to the germ of f at x.

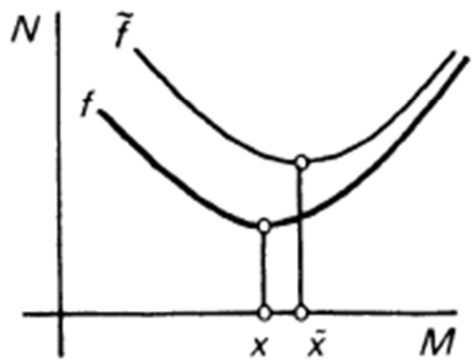


Fig. 6.