Network Theorems

OBJECTIVES

After studying this chapter you will be able to

- apply the superposition theorem to determine the current through or voltage across any resistance in a given network,
- state Thévenin's theorem and determine the Thévenin equivalent circuit of any resistive network,
- state Norton's theorem and determine the Norton equivalent circuit of any resistive network,
- determine the required load resistance of any circuit to ensure that the load receives maximum power from the circuit,
- apply Millman's theorem to determine the current through or voltage across any resistor supplied by any number of sources in parallel,
- state the reciprocity theorem and demonstrate that it applies for a given single-source circuit,
- state the substitution theorem and apply the theorem in simplifying the operation of a given circuit.

PUTTING IT IN PERSPECTIVE

André Marie Ampère

ANDRÉ MARIE AMPÈRE WAS BORN in Polémieux, Rhône, near Lyon, France on January 22, 1775. As a youth, Ampère was a brilliant mathematician who was able to master advanced mathematics by the age of twelve. However, the French Revolution, and the ensuing anarchy which swept through France from 1789 to 1799, did not exclude the Ampère family. Ampère's father, who was a prominent merchant and city official in Lyon, was executed under the guillotine in 1793. Young André suffered a nervous breakdown from which he never fully recovered. His suffering was further compounded in 1804, when after only five years of marriage, Ampère's wife died.

Even so, Ampère was able to make profound contributions to the field of mathematics, chemistry, and physics. As a young man, Ampère was appointed as professor of chemistry and physics in Bourg. Napoleon was a great supporter of Ampère's work, though Ampère had a reputation as an "absent-minded professor." Later he moved to Paris, where he taught mathematics.

9.1 Superposition Theorem

The **superposition theorem** is a method which allows us to determine the current through or the voltage across any resistor or branch in a network. The advantage of using this approach instead of mesh analysis or nodal analysis is that it is not necessary to use determinants or matrix algebra to analyze a given circuit. The theorem states the following:

The total current through or voltage across a resistor or branch may be determined by summing the effects due to each independent source.

In order to apply the superposition theorem it is necessary to remove all sources other than the one being examined. In order to "zero" a voltage source, we **replace it with a short circuit,** since the voltage across a short circuit is zero volts. A current source is zeroed by **replacing it with an open circuit,** since the current through an open circuit is zero amps.

If we wish to determine the power dissipated by any resistor, we must first find either the voltage across the resistor or the current through the resistor:

$$P = I^2R = \frac{V^2}{R}$$

EXAMPLE 9–1 Consider the circuit of Figure 9–1:

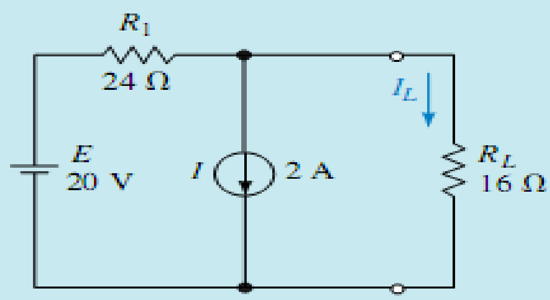


FIGURE 9–1

a. Determine the current in the load resistor, R_L.
b. Verify that the superposition theorem does not apply to power.

Solution

a. We first determine the current through R_L due to the voltage source by removing the current source and replacing in with an open circuit (zero amps) as shown in Figure 9–2.

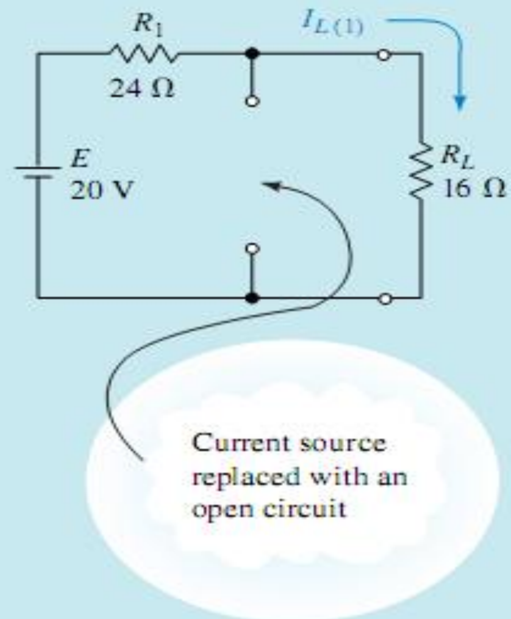
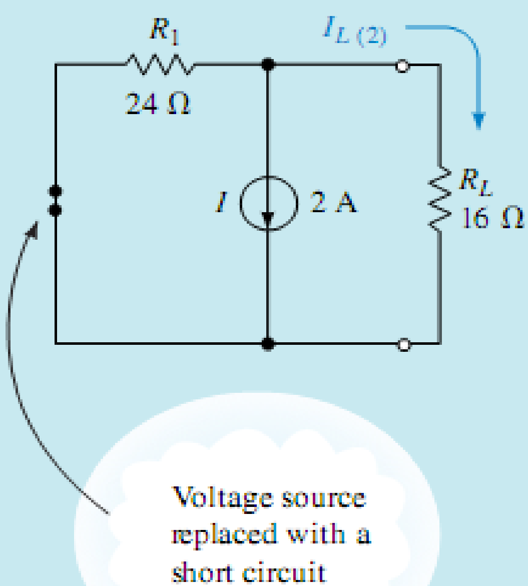


FIGURE 9–2

The resulting current through R_L is determined from Ohm's law as

$$I_{L(1)} = \frac{20\ \text{V}}{16\ \Omega + 24\ \Omega} = 0.500\ \text{A}$$

Next, we determine the current through R_L due to the current source by removing the voltage source and replacing it with a short circuit (zero volts) as shown in Figure 9–3.



The resulting current through R_L is found with the current divider rule as

$$I_{L(2)} = -\left(\frac{24\ \Omega}{24\ \Omega + 16\ \Omega}\right)(2\ \text{A}) = -1.20\ \text{A}$$

The resultant current through R_L is found by applying the superposition theorem:

$$I_L = 0.5\text{ A} - 1.2\text{ A} = -0.700\text{ A}$$

The negative sign indicates that the current through R_L is opposite to the assumed reference direction. Consequently, the current through R_L will, in fact, be upward with a magnitude of 0.7 A.

b. If we assume (incorrectly) that the superposition theorem applies for power, we would have the power due the first source given as

$$P_1 = I_{L(1)}^2 R_L = (0.5\text{ A})^2(16\ \Omega) = 4.0\text{ W}$$

and the power due the second source as

$$P_2 = I_{L(2)}^2 R_L = (1.2\text{ A})^2(16\ \Omega) = 23.04\text{ W}$$

The total power, if superposition applies, would be

$$P_T = P_1 + P_2 = 4.0\text{ W} + 23.04\text{ W} = 27.04\text{ W}$$

Clearly, this result is wrong, since the actual power dissipated by the load resistor is correctly given as

$$P_L = I_L^2 R_L = (0.7\text{ A})^2(16\ \Omega) = 7.84\text{ W}$$

9.2 Thévenin's Theorem

Any linear bilateral network may be reduced to a simplified two-terminal circuit consisting of a single voltage source in series with a single resistor as shown in Figure 9–9.

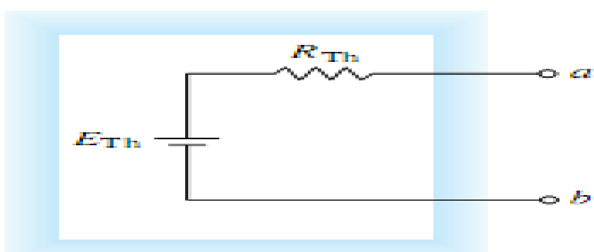


FIGURE 9–9 Thévenin equivalent circuit.

The following steps provide a technique which converts any circuit into its Thévenin equivalent:

1. Remove the load from the circuit.
2. Label the resulting two terminals. We will label them as *a* and *b*, although any notation may be used.
3. Set all sources in the circuit to zero.

   Voltage sources are set to zero by replacing them with short circuits (zero volts).

   Current sources are set to zero by replacing them with open circuits (zero amps).
4. Determine the Thévenin equivalent resistance, R_{Th}, by calculating the resistance "seen" between terminals *a* and *b*. It may be necessary to redraw the circuit to simplify this step.
5. Replace the sources removed in Step 3, and determine the open-circuit voltage between the terminals. If the circuit has more than one source, it may be necessary to use the superposition theorem. In that case, it will be necessary to determine the open-circuit voltage due to each source separately and then determine the combined effect. The resulting open-circuit voltage will be the value of the Thévenin voltage, E_{Th}.
6. Draw the Thévenin equivalent circuit using the resistance determined in Step 4 and the voltage calculated in Step 5. As part of the resulting circuit, include that portion of the network removed in Step 1.

EXAMPLE 9–3 Determine the Thévenin equivalent circuit external to the resistor R_L for the circuit of Figure 9–10. Use the Thévenin equivalent circuit to calculate the current through R_L.

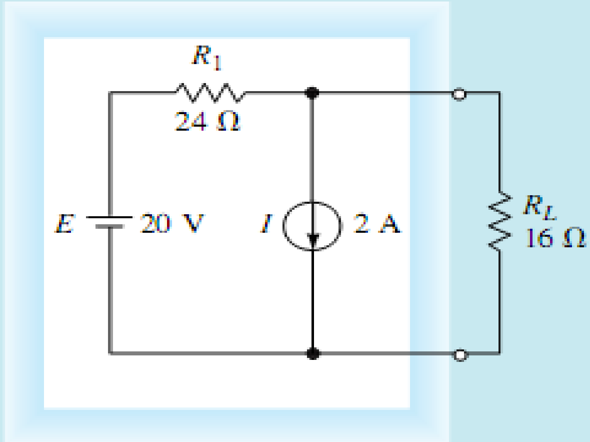


FIGURE 9–10

Solution

Steps 1 and 2: Removing the load resistor from the circuit and labelling the remaining terminals, we obtain the circuit shown in Figure 9–11.

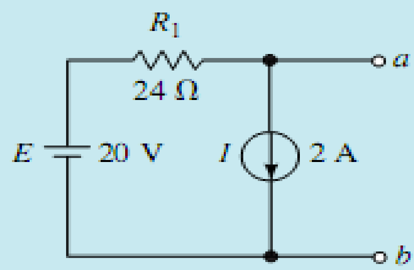
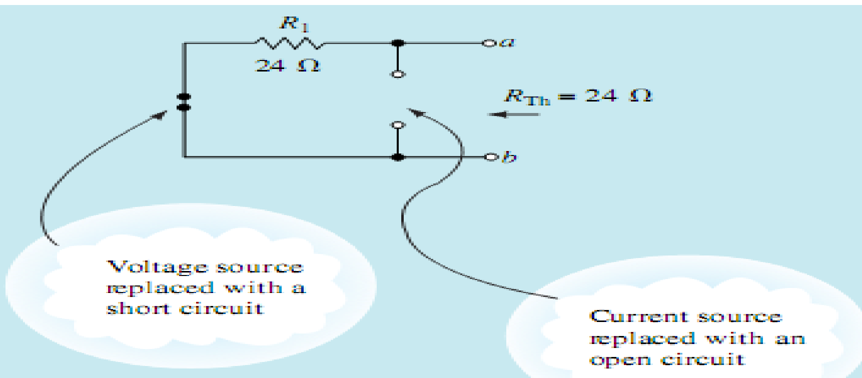


FIGURE 9–11

Step 3: Setting the sources to zero, we have the circuit shown in Figure 9–12.



Step 4: The Thévenin resistance between the terminals is $R_{Th} = 24\ \Omega$.

Step 5: From Figure 9–11, the open-circuit voltage between terminals a and b is found as

$$V_{ab} = 20\ \text{V} - (24\ \Omega)(2\ \text{A}) = -28.0\ \text{V}$$

Step 6: The resulting Thévenin equivalent circuit is shown in Figure 9–13.

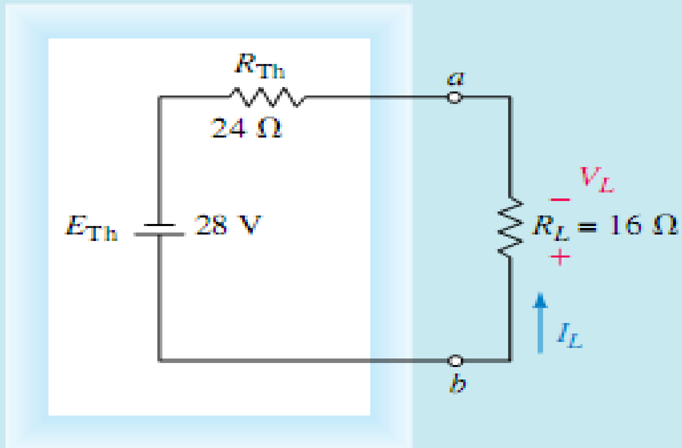


FIGURE 9–13

Using this Thévenin equivalent circuit, we easily find the current through R_L as

$$I_L = \left(\frac{28\ \text{V}}{24\ \Omega + 16\ \Omega}\right) = 0.700\ \text{A} \quad \text{(upward)}$$

9.3 Norton's Theorem

Norton's theorem is a circuit analysis technique which is similar to Thévenin's theorem. By using this theorem the circuit is reduced to a single current source and one parallel resistor. As with the Thévenin equivalent circuit, the resulting two-terminal circuit is equivalent to the original circuit when connected to any external branch or component. In summary, **Norton's theorem** may be simplified as follows:

Any linear bilateral network may be reduced to a simplified two-terminal circuit consisting of a single current source and a single shunt resistor as shown in Figure 9–26.

The following steps provide a technique which allows the conversion of any circuit into its Norton equivalent:

1. Remove the load from the circuit.
2. Label the resulting two terminals. We will label them as *a* and *b*, although any notation may be used.
3. Set all sources to zero. As before, voltage sources are set to zero by replacing them with short circuits and current sources are set to zero by replacing them with open circuits.
4. Determine the Norton equivalent resistance, R_N, by calculating the resistance seen between terminals *a* and *b*. It may be necessary to redraw the circuit to simplify this step.
5. Replace the sources removed in Step 3, and determine the current which would occur in a short if the short were connected between terminals *a* and *b*. If the original circuit has more than one source, it may be necessary to use the superposition theorem. In this case, it will be necessary to determine the short-circuit current due to each source separately and then determine the combined effect. The resulting short-circuit current will be the value of the Norton current I_N.
6. Sketch the Norton equivalent circuit using the resistance determined in Step 4 and the current calculated in Step 5. As part of the resulting circuit, include that portion of the network removed in Step 1.

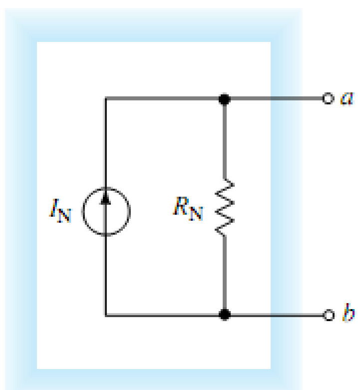


FIGURE 9–26 Norton equivalent circuit.

EXAMPLE 9–7 Find the Norton equivalent of the circuit external to resistor R_L in the circuit in Figure 9–34. Use the equivalent circuit to determine the load current I_L when $R_L = 0$, 2 kΩ, and 5 kΩ.

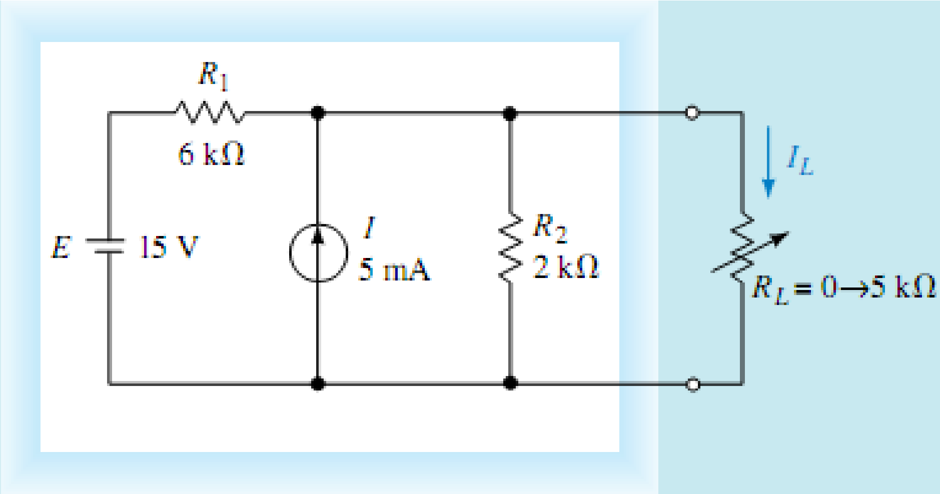


FIGURE 9–34

Solution

Steps 1, 2, and 3: After removing the load resistor, labelling the remaining two terminals *a* and *b*, and setting the sources to zero, we have the circuit of Figure 9–35.

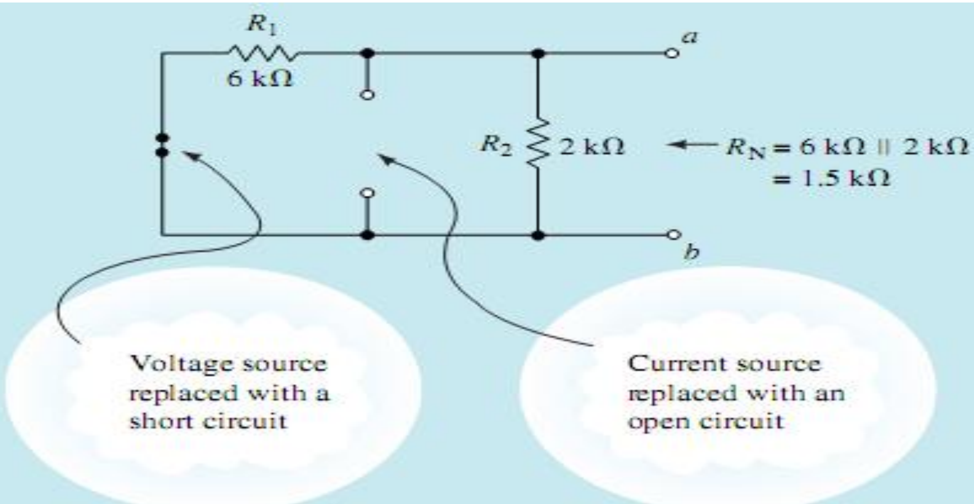


FIGURE 9–35

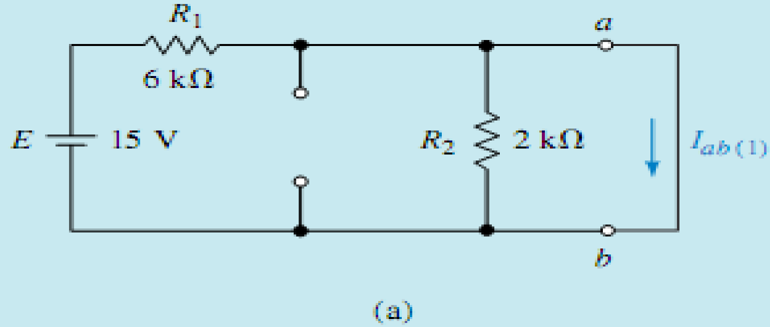
Step 4: The Norton resistance of the circuit is found as

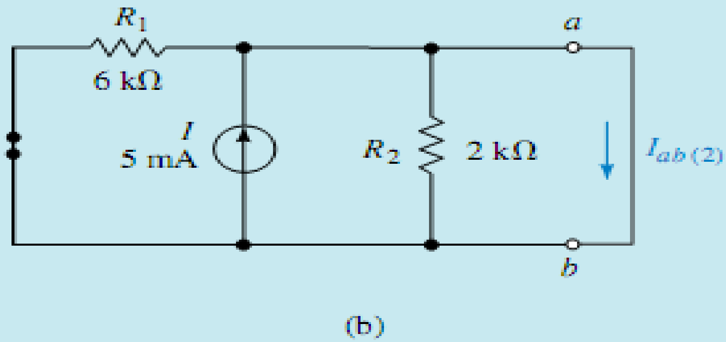
$$R_N = 6\text{ k}\Omega \| 2\text{ k}\Omega = 1.5\text{ k}\Omega$$

Step 5: The value of the Norton constant-current source is found by determining the current effects due to each independent source acting on a short circuit between terminals *a* and *b*.

Voltage Source, E: Referring to Figure 9–36(a), a short circuit between terminals *a* and *b* eliminates resistor R_2 from the circuit. The short-circuit current due to the voltage source is

$$I_{ab(1)} = \frac{15\text{ V}}{6\text{ k}\Omega} = 2.50\text{ mA}$$





Current Source, I: Referring to Figure 9–36(b), the short circuit between terminals *a* and *b* eliminates both resistors R_1 and R_2. The short-circuit current due to the current source is therefore

$$I_{ab(2)} = 5.00\text{ mA}$$

The resultant Norton current is found from superposition as

$$I_N = I_{ab(1)} + I_{ab(2)} = 2.50\text{ mA} + 5.00\text{ mA} = 7.50\text{ mA}$$

Step 6: The Norton equivalent circuit is shown in Figure 9–37.

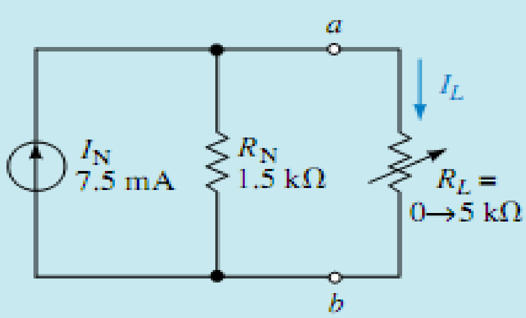


FIGURE 9–37

Let $R_L = 0$*:* The current I_L must equal the source current, and so

$$I_L = 7.50\text{ mA}$$

Let $R_L = 2\text{ k}\Omega$*:* The current I_L is found from the current divider rule as

$$I_L = \left(\frac{1.5\text{ k}\Omega}{1.5\text{ k}\Omega + 2\text{ k}\Omega}\right)(7.50\text{ mA}) = 3.21\text{ mA}$$

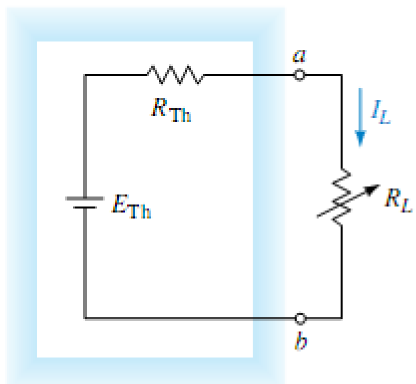
Let $R_L = 5\ k\Omega$: Using the current divider rule again, the current I_L is found as

$$I_L = \left(\frac{1.5\ k\Omega}{1.5\ k\Omega + 5\ k\Omega}\right)(7.50\ mA) = 1.73\ mA$$

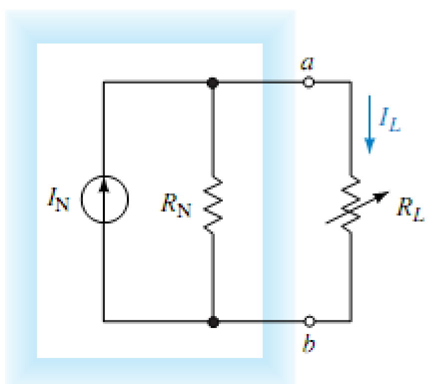
9.4 Maximum Power Transfer Theorem

The **maximum power transfer theorem** states the following:

A load resistance will receive maximum power from a circuit when the resistance of the load is exactly the same as the Thévenin (Norton) resistance looking back at the circuit.



(a)



(b)

$$R_L = R_{Th} = R_N$$

EXAMPLE 9–10 Consider the circuit of Figure 9–50:

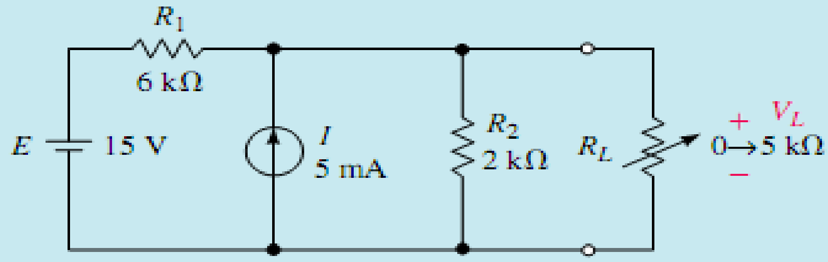


FIGURE 9–50

a. Determine the value of load resistance required to ensure that maximum power is transferred to the load.
b. Find V_L, I_L, and P_L when maximum power is delivered to the load.

Solution

a. In order to determine the conditions for maximum power transfer, it is first necessary to determine the equivalent circuit external to the load. We may determine either the Thévenin equivalent circuit or the Norton equivalent circuit. This circuit was analyzed in Example 9–4 using Thévenin's theorem, and we determined the equivalent circuit to be as shown in Figure 9–51.

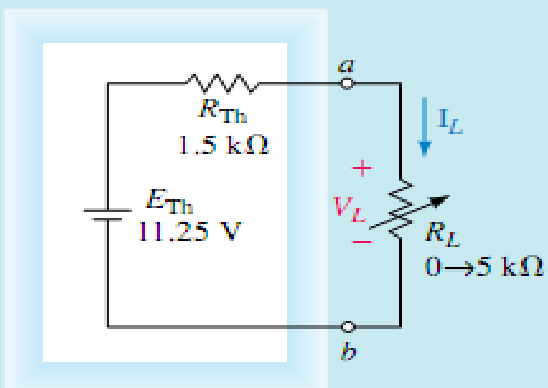


FIGURE 9–51

Maximum power will be transferred to the load when $R_L = 1.5\ \text{k}\Omega$.

b. Letting $R_L = 1.5\ \text{k}\Omega$, we see that half of the Thévenin voltage will appear across the load resistor and half will appear across the Thévenin resistance. So, at maximum power,

$$V_L = \frac{E_{\text{Th}}}{2} = \frac{11.25\ \text{V}}{2} = 5.625\ \text{V}$$

$$I_L = \frac{5.625\ \text{V}}{1.5\ \text{k}\Omega} = 3.750\ \text{mA}$$

The power delivered to the load is found as

$$P_L = \frac{V_L^2}{R_L} = \frac{(5.625\ \text{V})^2}{1.5\ \text{k}\Omega} = 21.1\ \text{mW}$$

Or, alternatively using current, we calculate the power as

$$P_L = I_L^2 R_L = (3.75\ \text{mA})^2(1.5\ \text{k}\Omega) = 21.1\ \text{mW}$$

In solving this problem, we could just as easily have used the Norton equivalent circuit to determine required values.