Numerical integration:

When we cannot evaluate a definite integral$\left[\int_a^b f(x)dx\right]$, we turn to numerical methods.

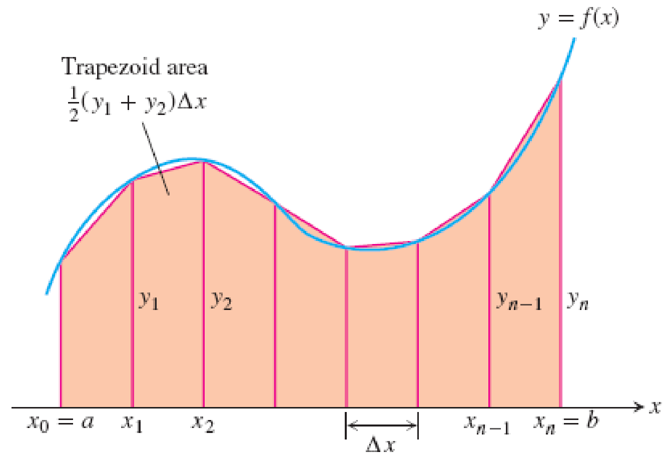
a) The Trapezoidal Rule

to approximate $\int_a^b f(x)dx$

use

$$T = \frac{h}{2}[y_o + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n]$$

$n =$ subinterval of length $h = \dfrac{b-a}{n}$



Example: use trapezoidal rule with $n = 4$ to estimate $\displaystyle\int_1^2 x^2\,dx$

Then, compare the estimate with the exact value of integral

Sol.: The exact value of the integral is:

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\bigg|_1^2 = \frac{1}{3}[2^3 - 1^3] = \frac{1}{3}[8-1] = \frac{7}{3} = 2.3333$$

To find the trapezoidal approximation, we divide the interval of integration onto (n = 4) subintervals of equal length and list the values of $y = x^2$ at the endpoints and subdivision points, as following:-

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

X		$y=x^2$	
$x_o=a$	1	y_o	1
$x_1=a+h$	5/4	y_1	25 / 16
$x_2=a+2h$	6/4	y_2	36 / 16
$x_3=a+3h$	7/4	y_3	49 / 16
$x_4=a+4h=b$	2	y_4	4

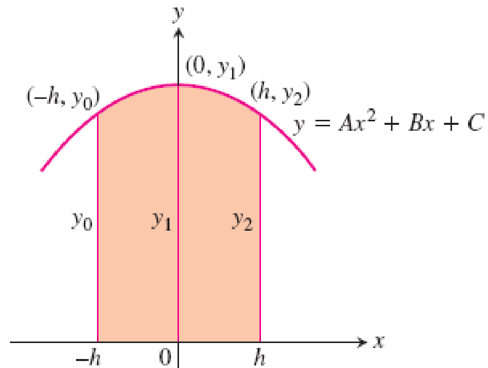
Use $T = \frac{h}{2}[y_o + 2y_1 + 2y_2 + 2y_3 + y_4]$

$$T = \frac{1/4}{2}\left[1 + 2\frac{25}{16} + 2\frac{36}{16} + 2\frac{49}{16} + 4\right] = \frac{75}{32} = 2.34375$$

b) Simpson Rule

Simpson's rule is based on approximating curves with parabolas instead of line segments. The shaded area under the parabola is:

$$A_p = \frac{h}{3}(y_o + 4y_1 + y_2)$$



By integrating from –*h* to *h*, we find the shaded area to be $\frac{h}{3}(y_0 + 4y_1 + y_2)$

Applying this formula successively along continuous curve *y=f(x)* from *x=a* to *x=b* leads to an estimate of $\int_a^b f(x)dx$ that is more accurate than T for a given step size h.

To approximate $\int_a^b f(x)dx$ use:

$$S = \frac{h}{3}[y_o + 4y_1 + 2y_2 + 4y_3 \ldots + 2y_{n-2} + 4y_{n-1} + y_n]$$

Where $h = \frac{b-a}{n}$ and n is even

Examples: Use Simpson's rule to approximate:

1) $\int_1^2 x^2\, dx$ use $n = 4$

Sol.:

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

$$S = \frac{h}{3}[y_o + 4y_1 + 2y_2 + 4y_3 \ldots + 2y_{n-2} + 4y_{n-1} + y_n]$$

$$S=\frac{\left(\frac{1}{4}\right)}{3}\left[1+4\left(\frac{25}{16}\right)+2\left(\frac{36}{16}\right)+4\left(\frac{49}{16}\right)+4\right]$$

$$=\frac{1}{12}\left[\frac{8+50+36+98+32}{8}\right]=\frac{7}{3}=2.333$$

2) $\int_0^1 5x^4dx$ use $n=4$

x	$y=5x^4$
0	0
1/4	5/256
2/4	80/256
3/4	405/256
1	5

Exact Solution:

$$\int_0^1 5x^4dx = x^5 \,|_0^1 = 1$$

Using Simpson's rule

$$h=\frac{b-a}{n}=\frac{1-0}{4}=\frac{1}{4}$$

$$S=\frac{h}{3}[y_o+4y_1+2y_2+4y_3+y_4]$$

$$S=\frac{1}{12}\left[0+4*\frac{5}{256}+2*\frac{80}{256}+4*\frac{405}{256}+5\right]=1.0026$$

Improper Integrals

Definition $\int_a^b f(x)dx=$ denotes an improper integral if:

1) $f(x)$ becomes infinite at one or more points of the interval of integration .

2) One or both of the limits of the integral is infinite.

3) Both (1) and (2) hold.

Definitions:

1) If $f(x)$ is continuous on the interval $[a,b]$ except for infinite discontinuities at

i) Right-end point b, then:

$$\int_a^b f(x)dx=\lim_{l\to b^-}\int_a^l f(x)dx$$

ii) left-end point a, then:

$$\int_a^b f(x)dx = \lim_{l \to a^+} \int_l^b f(x)dx$$

iii) Interior point c where $a<c<b$, then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

2) The improper integral of f over the interval:

i) (-∞,b], then:

$$\int_{-\infty}^b f(x)dx = \lim_{l \to -\infty} \int_l^b f(x)dx$$

ii) [a,∞), then:

$$\int_a^\infty f(x)dx = \lim_{l \to \infty} \int_a^l f(x)dx$$

ii) (-∞,∞) ,then:

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx \quad \text{where } c \in R$$

- In the case where the limit exists, the improper integral is said to converge, and the limit is defined to be the value of the integral.
- In the case where the limit does not exist, the improper integral is said to diverge, and it is not assigned a value.

Example1: Determine whether the integral:

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx \ \text{ converge or diverge}$$

Sol.: because of $f(x)$ becomes infinite at $x=1$, so it is an improper integral, so:

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx = \lim_{l \to 1^-} \int_0^l \sqrt{\frac{1+x}{1-x}}\, dx$$

Multiply denominator and numerator by $\sqrt{1+x}$ gives:

$$\lim_{l\to1^-}\int_0^l \sqrt{\frac{1+x}{1-x}}\,dx * \left(\frac{\sqrt{1+x}}{\sqrt{1+x}}\right) = \lim_{l\to1^-}\int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$\lim_{l\to1^-}\left(\int_0^l \frac{dx}{\sqrt{1-x^2}} + \int_0^l \frac{xdx}{\sqrt{1-x^2}}\right) = \lim_{l\to1^-}\left(\sin^{-1}x - \sqrt{1-x^2}\right)\Big]_0^l$$

$$\lim_{l\to1^-}\left[\left(\sin^{-1}l - \sqrt{1-l^2}\right) - \left(\sin^{-1}0 - \sqrt{1}\right)\right]$$

$$\lim_{l\to1^-}\left[\left(\sin^{-1}l - \sqrt{1-l^2}\right) + 1\right] = \sin^{-1}1 - 0 + 1 = \frac{\pi}{2} + 1$$

So the integral converges to $\pi/2+1$.

Example 2: evaluate the following:

a) $\int_0^1 \frac{dx}{x}$

the function $\frac{1}{x}$ becomes infinite at x=0 , thus it is an improper integral

so, $\int_0^1 \frac{dx}{x} = \lim_{l\to0^+}\int_l^1 \frac{dx}{x} = \lim_{l\to0^+}[\ln x]_l^1 = \lim_{l\to0^+}[\ln 1 - \ln l] = \lim_{l\to0^+}\left[\ln\frac{1}{l}\right] = +\infty$

so the integral $\int_0^1 \frac{dx}{x}$ diverges

b) $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

Sol.: the function $\frac{1}{(x-1)^{2/3}}$ becomes infinite at x=1 $\in$[0,3] ,so it is an improper integral

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{l\to1^-}\int_0^l \frac{dx}{(x-1)^{2/3}} + \lim_{k\to1^+}\int_k^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{l\to1^-}\left[\frac{(x-1)^{1/3}}{1/3}\right]_0^l + \lim_{k\to1^+}\left[\frac{(x-1)^{1/3}}{1/3}\right]_k^3$$

$$= \lim_{l\to1^-}\left[3(l-1)^{1/3} - 3(-1)^{1/3}\right] + \lim_{k\to1^+}\left[3(2)^{1/3} - 3(k-1)^{1/3}\right]$$

$$\to \lim_{l\to1^-}\left[3(l-1)^{1/3} - 3(-1)^{1/3}\right] = 3 \text{ converges}$$

and $\lim_{k\to1^+}\left[3(2)^{1/3} - 3(k-1)^{1/3}\right] = 3\sqrt[3]{2}$ conerges too since both limits exist and finite, the integral of f converges and its value is $(3+3\sqrt[3]{2})$

The integral $\int_1^\infty \frac{dx}{x^p}$

Note: the convergence of the integral $\int_1^\infty \frac{dx}{x^p}$ depends on the value of the exponent. This will be illustrated in these examples.

Example: Do the integrals $\int_1^\infty \frac{dx}{x}$ and $\int_1^\infty \frac{dx}{x^2}$ converge or divrege?

Sol.: a) $\int_1^\infty \frac{dx}{x} = \lim_{l\to\infty}\int_1^l \frac{dx}{x} = \lim_{l\to\infty}[\ln x]_1^l = \lim_{l\to\infty}(\ln l - \ln 1) = \lim_{l\to\infty}(\ln l) = \infty$

so the integral $\int_1^\infty \frac{dx}{x}$ diverges

b) $\int_1^\infty \frac{dx}{x^2} = \lim_{l\to\infty}\int_1^l \frac{dx}{x^2} = \lim_{l\to\infty}\left[\frac{-1}{x}\right]_1^l = \lim_{l\to\infty}\left[\frac{-1}{l} - \left(\frac{-1}{1}\right)\right] = \lim_{l\to\infty}\left[\frac{-1}{l} + 1\right]$

so the integral $\int_1^\infty \frac{dx}{x^2}$ converges and it's value is 1.

Generally , the integral $\int_1^\infty \frac{dx}{x^p}$

a) diverges when $p \leq 1$

b)converges when $p > 1$ and it's value is $\frac{1}{p-1}$

Example: Evaluate the following:

1) $\int_0^\infty (1-x)\, e^{-x} dx$

Sol.: $\int (1-x)e^{-x}dx$ by parts

Let $u = (1-x) \quad \to dv = e^{-x}dx$

$du = -dx \quad \to v = -e^{-x}$

$$\int (1-x)e^{-x}dx = -e^{-x}(1-x) - \int e^{-x}dx = -e^{-x}(1-x) + e^{-x} + C$$

$$= -\cancel{e^{-x}} + xe^{-x} + \cancel{e^{-x}} + C = xe^{-x} + C$$

$$\therefore \int_0^{\infty} (1-x)\, e^{-x} dx = \lim_{l\to\infty} \int_0^{l} (1-x)\, e^{-x} dx = \lim_{l\to\infty} xe^{-x}\Big]_0^{l} = \lim_{l\to\infty}[le^{-l} - 0 * l]$$

$$= \lim_{l\to\infty} \frac{l}{e^{l}} = \frac{\infty}{\infty}$$

By L'Hopital's rule:

$$\lim_{l\to\infty} \frac{l}{e^{l}} = \lim_{l\to\infty} \frac{1}{e^{l}} = 0$$

So the integral $\int_0^{\infty}(1-x)\, e^{-x} dx$ converges and its value is (0).

2) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$ use $c = 0$

Sol.:

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\therefore \int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{l\to-\infty} \int_l^{0} \frac{dx}{1+x^2} = \lim_{l\to-\infty} \tan^{-1} x\Big]_l^{0} = \lim_{l\to-\infty} (\tan^{-1} 0 - \tan^{-1} l)$$

$$= \lim_{l\to\infty}(-\tan^{-1} l) = \frac{\pi}{2} \quad \text{converges}$$

$$\therefore \int_0^{\infty} \frac{dx}{1+x^2} = \lim_{k\to\infty} \int_0^{k} \frac{dx}{1+x^2} = \lim_{k\to\infty} \tan^{-1} x\Big]_0^{k} = \lim_{k\to\infty} (\tan^{-1} k - \tan^{-1} 0)$$

$$= \lim_{k\to\infty} (\tan^{-1} k) = \frac{\pi}{2} \qquad \text{converges}$$

Thus, the integral converges and its value is $\pi = (\frac{\pi}{2} + \frac{\pi}{2})$

3) $\int_0^{\infty} \frac{dx}{\sqrt{x}(x-1)}$

This integral is improper for two reasons:

1) The interval of the integration is infinite

2) There is discontinuity at $x=0$

To evaluate this integral we will split the integral of integration at a convenient point, say $x=1$ and write :

$$\int_0^\infty \frac{dx}{\sqrt{x}(x-1)} = \int_0^1 \frac{dx}{\sqrt{x}(x-1)} + \int_1^\infty \frac{dx}{\sqrt{x}(x-1)}$$

To evaluate $\int \frac{dx}{\sqrt{x}(x-1)}$ $assume\ z^2 = x \ \rightarrow \therefore dx = 2zdz$

$$\int \frac{dx}{\sqrt{x}(x-1)} = \int \frac{2zdz}{z(z^2+1)} = 2\int \frac{dz}{z^2+1} = 2\tan^{-1} z + C = 2\tan^{-1}\sqrt{x} + C$$

$$\int_0^1 \frac{dx}{\sqrt{x}(x-1)} = \lim_{l\to 0}\int_l^1 \frac{dx}{\sqrt{x}(x-1)} = \lim_{l\to 0}\left[2\tan^{-1}\sqrt{x}\right]_l^1$$

$$= 2\lim_{l\to 0}\left[\tan^{-1} 1 - \tan^{-1}\sqrt{l}\right] = 2\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{2} \qquad \text{converges}$$

$$\int_1^\infty \frac{dx}{\sqrt{x}(x-1)} = \lim_{k\to\infty}\int_1^k \frac{dx}{\sqrt{x}(x-1)} = \lim_{k\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^k$$

$$= 2\lim_{k\to\infty}\left[\tan^{-1}\sqrt{k} - \tan^{-1} 1\right] = 2\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{2} \qquad \text{converges}$$

Thus, the integral converges and its value is $\pi = \left(\frac{\pi}{2} + \frac{\pi}{2}\right)$