The substitution $z = \tan\frac{x}{2}$

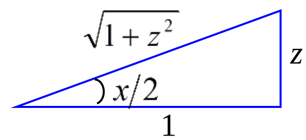
The substitution $z = \tan\frac{x}{2}$ reduce the problem of integrating any rational function of $\sin x$ and $\cos x$ into a problem involving a rational function of z .

When $z = \tan(x/2) \Rightarrow \frac{x}{2} = \tan^{-1} z \Rightarrow x = 2\tan^{-1} z \Rightarrow dx = 2.\left(\frac{dz}{1+z^2}\right)$

$$\boxed{dx = \frac{2dz}{1+z^2}}$$

From the figure you can see that:

$\sin(x/2) = \frac{z}{\sqrt{1+z^2}}$ and $\cos(x/2) = \frac{1}{\sqrt{1+z^2}}$



You know that:

$\sin 2\theta = 2\sin\theta\cos\theta$

And $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Assume $x = 2\theta \Rightarrow \theta = \frac{x}{2}$

$\therefore \sin x = 2\sin(x/2)\cos(x/2)$

$$\sin x = 2\left(\frac{z}{\sqrt{1+z^2}}\right)\left(\frac{1}{\sqrt{1+z^2}}\right)$$

$$\therefore \boxed{\sin x = \frac{2z}{1+z^2}}$$

And $\cos x = \cos^2(x/2) - \sin^2(x/2)$

$$\cos x = \left(\frac{1}{\sqrt{1+z^2}}\right)^2 - \left(\frac{z}{\sqrt{1+z^2}}\right)^2 = \frac{1}{1+z^2} - \frac{z^2}{1+z^2}$$

$$\therefore \boxed{\cos x = \frac{1-z^2}{1+z^2}}$$

Example 2: Evaluate the following:

1) $\int \frac{dx}{2+\sin x}$

Sol.:

Let $z = \tan\frac{x}{2} \rightarrow dx = \frac{2dz}{1+z^2}$ *and* $\sin x = \frac{2z}{1+z^2}$

So, $\int \frac{\frac{2dz}{1+z^2}}{2+\frac{2z}{1+z^2}} = \int \frac{\frac{2dz}{1+z^2}}{\frac{2(1+z^2)+2z}{1+z^2}} = \int \frac{dz}{1+z+z^2}$

$$= \int \frac{dz}{z^2+z+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1} = \int \frac{dz}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Let $u = z+\frac{1}{2} \quad \rightarrow du = dz$

$\therefore \int \frac{du}{u^2+\frac{3}{4}} = \int \frac{du}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)} = \frac{4}{3}\int \frac{du}{\left(\frac{2}{\sqrt{3}}u\right)^2+1}$

Let $\theta = \frac{2}{\sqrt{3}}u \quad \rightarrow d\theta = \frac{2}{\sqrt{3}}du \;\; or \; du = \frac{\sqrt{3}}{2}\, d\theta$

$= \frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}\, d\theta}{\theta^2+1} = \frac{2}{\sqrt{3}}\int \frac{d\theta}{\theta^2+1} = \frac{2}{\sqrt{3}}\tan^{-1}\theta + C$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(z+\frac{1}{2}\right)\right) + C$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right) + c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right) + C$

2) $\int \frac{dx}{1-\sin x+\cos x}$

Sol.:

Let $z = \tan\frac{x}{2} \rightarrow dx = \frac{2\,dz}{1+z^2} \rightarrow \sin x = \frac{2z}{1+z^2} \;\; and \;\; \cos x = \frac{1-z^2}{1+z^2}$

So, $\int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}} = \int \frac{\frac{2\,dz}{1+z^2}}{\frac{1+z^2-2z+1-z^2}{1+z^2}} = \int \frac{2\,dz}{2-2z} = \int \frac{dz}{1-z}$

$$= -\ln|1-z| + c == -\ln\left|1-\tan\frac{x}{2}\right| + C$$