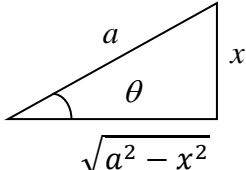
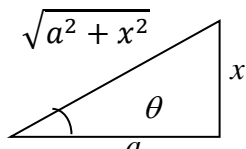
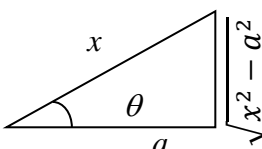


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Trigonometric Substitutions:

We will concern with integrals contain expressions of the form:

$a^2 + x^2$, $a^2 - x^2$ and $x^2 - a^2$ where x is variable and a is constant.

| Expression in the integrand | Substitution | Restriction on θ | Simplification | Reference triangle |
|-----------------------------|---|---|--|--|
| $a^2 - x^2$ | $x = a \sin \theta$ $dx = a \cos \theta d\theta$ | $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ | $a^2 - x^2 = a^2 - a^2 \sin^2 \theta$ $= a^2(1 - \sin^2 \theta)$ $= a^2 \cos^2 \theta$ |  |
| $a^2 + x^2$ | $x = a \tan \theta$ $dx = a \sec^2 \theta d\theta$ | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ | $a^2 + x^2 = a^2 + a^2 \tan^2 \theta$ $= a^2(1 + \tan^2 \theta)$ $= a^2 \sec^2 \theta$ |  |
| $x^2 - a^2$ | $x = a \sec \theta$ $dx = a \sec \theta \tan \theta d\theta$ | $0 \leq \theta < \frac{\pi}{2}$ if $\frac{x}{a} \geq 1$ $\frac{\pi}{2} < \theta \leq \pi$ if $\frac{x}{a} \leq -1$ | $x^2 - a^2 = a^2 \sec^2 \theta - a^2$ $= a^2(\sec^2 \theta - 1)$ $= a^2 \tan^2 \theta$ |  |

Examples: Evaluate the following:

1) $\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{dx}{\sqrt{2^2+x^2}}$

Sol.: Let $x = 2 \tan \theta \rightarrow dx = 2 \sec^2 \theta d\theta$

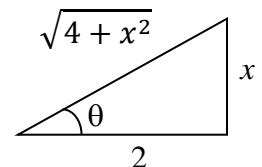
And $4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta$

$= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4 \sec^2 \theta}} = \int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C$

$= \ln|\sec \theta + \tan \theta| + c$

$= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C = \ln \left| \left(\sqrt{4+x^2} + x \right) \right| - \ln 2 + C$

$= \ln \left| \left(\sqrt{4+x^2} + x \right) \right| + C_1$ where $C_1 = C - \ln 2$



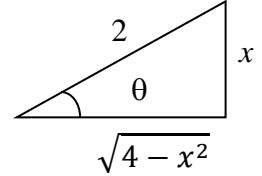
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$$2) \int_1^{\sqrt{2}} \frac{dx}{x^2 \sqrt{4-x^2}}$$

Sol.: Let $x = 2 \sin \theta \rightarrow dx = 2 \cos \theta d\theta$

$$\text{and } 4 - x^2 = 4 - 4 \sin^2 \theta = 4(1 - \sin^2 \theta) = 4 \cos^2 \theta$$

$$\begin{aligned} \text{so, } \int \frac{dx}{x^2 \sqrt{4-x^2}} &= \int \frac{2 \cos \theta d\theta}{4 \sin^2 \theta * \sqrt{4 \cos^2 \theta}} = \frac{1}{4} \int \frac{d\theta}{\sin^2 \theta} \\ &= \frac{1}{4} \int \csc^2 \theta d\theta = -\frac{1}{4} \cot \theta + c = -\frac{1}{4} \frac{\sqrt{4-x^2}}{x} + C \end{aligned}$$



and

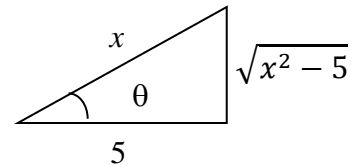
$$\int_1^{\sqrt{2}} \frac{dx}{x^2 \sqrt{4-x^2}} = \frac{-1}{4} \left[\frac{\sqrt{4-x^2}}{x} \right]_1^{\sqrt{2}} = -\frac{1}{4} \left[\frac{\sqrt{4-2}}{\sqrt{2}} - \frac{\sqrt{4-1}}{1} \right] = \frac{\sqrt{3}-1}{4}$$

$$3) \int \frac{dx}{\sqrt{x^2-25}}$$

Sol.: Let $x = 5 \sec \theta \rightarrow dx = 5 \sec \theta \tan \theta d\theta$

$$\text{and } x^2 - 25 = 25 \sec^2 \theta - 25 = 25(\sec^2 \theta - 1) = 25 \tan^2 \theta$$

$$\begin{aligned} &= \int \frac{5 \sec \theta \tan \theta d\theta}{\sqrt{25 \tan^2 \theta}} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{x}{5} + \frac{\sqrt{x^2-25}}{5} \right| + C \end{aligned}$$



Integrals involving $ax^2 + bx + c = 0$, $a \neq 0$

We handle these first by completing the squares

$$4) \int \frac{dx}{\sqrt{2x-x^2}}$$

Sol.:

$$2x - x^2 = -(x^2 - 2x + 1) + 1 = 1 - (x-1)^2$$

Then substitute $u = (x-1) = du = dx$

$$\text{So, } \int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(x-1) + C$$

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$$5) \int \frac{dx}{\sqrt{x^2 + 2x}} = \int \frac{dx}{\sqrt{x^2 + 2x + 1 - 1}} = \int \frac{dx}{\sqrt{(x+1)^2 - 1}}$$

Sol.: Let $u = (x+1) \rightarrow du = dx$

$$= \int \frac{du}{\sqrt{u^2 - 1}}$$

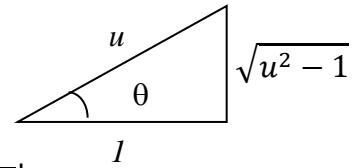
Let $u = \sec \theta \rightarrow du = \sec \theta \tan \theta d\theta$

and $u^2 - 1 = \sec^2 \theta - 1 = \tan^2 \theta$

$$= \int \frac{\sec \theta \tan \theta d\theta}{\tan \theta} = \int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C$$

$$= \ln \left| \frac{u}{1} + \frac{\sqrt{u^2 - 1}}{1} \right| + c = \ln \left| (x+1) + \sqrt{(x+1)^2 - 1} \right| + C$$

$$= \ln \left| (x+1) + \sqrt{x^2 + 2x + 1 - 1} \right| + C = \ln \left| (x+1) + \sqrt{x^2 + 2x} \right| + C$$



$$6) \int \frac{xdx}{x^2 - 4x + 8} = \int \frac{xdx}{x^2 - 4x + 4 + 4} = \int \frac{xdx}{(x-2)^2 + 4}$$

Let $u = x - 2 \rightarrow du = dx$ and $x = u + 2$

$$= \int \frac{(u+2)du}{u^2 + 4} = \int \frac{u du}{u^2 + 4} + \int \frac{2 du}{u^2 + 4}$$

$$= \frac{1}{2} \ln|u^2 + 4| + 2 \left(\frac{1}{2} \right) \tan^{-1} \left(\frac{u}{2} \right) + C$$

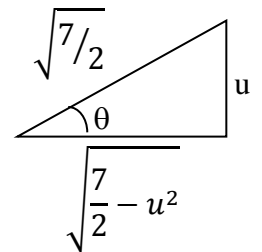
$$= \frac{1}{2} \ln|(x-2)^2 + 4| + 2 \left(\frac{1}{2} \right) \tan^{-1} \left(\frac{x-2}{2} \right) + C$$

$$7) \int \frac{dx}{\sqrt{5 - 4x - 2x^2}} = \int \frac{dx}{\sqrt{5 - 2(x^2 + 2x + 1 - 1)}} \\ = \int \frac{dx}{\sqrt{5 - 2(x+1)^2 + 2}} = \int \frac{dx}{\sqrt{7 - 2(x+1)^2}}$$

Let $u = x + 1 \rightarrow du = dx$

$$= \int \frac{du}{\sqrt{7 - 2u^2}} = \int \frac{du}{\sqrt{2 \left(\frac{7}{2} - u^2 \right)}} = \frac{1}{\sqrt{2}} \int \frac{du}{\sqrt{\frac{7}{2} - u^2}}$$

$$\therefore u = \sqrt{\frac{7}{2}} \sin \theta \rightarrow du = \sqrt{\frac{7}{2}} \cos \theta d\theta \rightarrow \frac{7}{2} - u^2 = \frac{7}{2} - \frac{7}{2} \sin^2 \theta = \frac{7}{2} \cos^2 \theta$$



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$$\begin{aligned} &= \frac{1}{\sqrt{2}} \int \frac{\left(\sqrt{\frac{7}{2}}\right) \cos \theta \, d\theta}{\sqrt{\frac{7}{2}} \cos^2 \theta} = \frac{1}{\sqrt{2}} \int d\theta = \frac{1}{\sqrt{2}} \theta + C = \frac{1}{\sqrt{2}} \sin^{-1} \left(\sqrt{\frac{2}{7}} u \right) + C \\ &= \frac{1}{\sqrt{2}} \sin^{-1} \left(\sqrt{\frac{2}{7}} (x+1) \right) + C \end{aligned}$$

Homework: Evaluate the following integrals;

- 1) $\int \sqrt{4-x^2} \, dx$ 2) $\int \sqrt{1-4x^2} \, dx$ 3) $\int \frac{x^2 dx}{\sqrt{9-x^2}}$ 4) $\int \frac{dx}{x^2 \sqrt{16-x^2}}$
- 5) $\int \frac{dx}{(4-x^2)^2}$ 6) $\int \frac{x^2 dx}{\sqrt{5+x^2}}$ 7) $\int \frac{\sqrt{x^2-9} \, dx}{x}$ 8) $\int x^3 \sqrt{5-x^2} \, dx$
- 9) $\int \frac{dx}{(1-x^2)^{3/2}}$ 10) $\int \frac{x^3 dx}{\sqrt{2-x^2}}$ 11) $\int e^x \sqrt{1-e^{2x}} \, dx$ 12) $\int \frac{\cos \theta \, d\theta}{\sqrt{2-\sin^2 \theta}}$
- 13) $\int x^3 \sqrt{16-x^2} \, dx$ 14) $\int_0^{1/3} \frac{dx}{(4-9x^2)^2}$ 15) $\int_0^3 \frac{x^3 dx}{(3+x^2)^{5/2}}$ 16) $\int \frac{dx}{1+2x^2+x^4}$
- 17) $\int \frac{dx}{x^2-4x+13}$ 18) $\int \frac{dx}{\sqrt{2x-x^2}}$ 19) $\int \frac{e^x dx}{\sqrt{1+e^x+e^{2x}}}$ 20) $\int_0^1 \sqrt{x(4-x)} \, dx$
- 21) $\int (x \cos x + \sin x) \sqrt{1+x^2 \sin^2 x} \, dx$ let $u = x \sin x$
- 22) $\int \sin x \cos x \sqrt{1-\sin^4 x} \, dx$ let $u = \sin^2 x$

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Mathematics