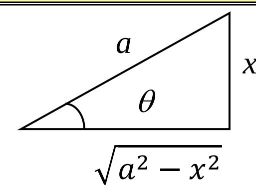
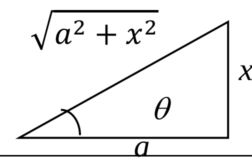
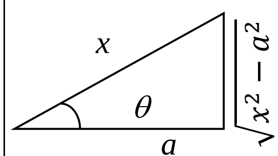
syllabus

Trigonometric Substitutions:

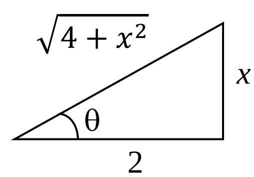
We will concern with integrals contain expressions of the form:

$a^2 + x^2, a^2 - x^2$ and $x^2 - a^2$ where x is variable and a is constant.

Expression in the integrand	Substitution	Restriction on θ	Simplification	Reference triangle
$a^2 - x^2$	$x = a\sin\theta$ $dx = a\cos\theta d\theta$	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$	$a^2 - x^2 = a^2 - a^2\sin^2\theta$ $=a^2(1-\sin^2\theta)$ $=a^2\cos^2\theta$	
$a^2 + x^2$	$x = a\tan\theta$ $dx = a\sec^2\theta d\theta$	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + x^2 = a^2 + a^2\tan^2\theta$ $=a^2(1+\tan^2\theta)$ $=a^2\sec^2\theta$	
$x^2 - a^2$	$x = a\sec\theta\, dx$ $dx = a\sec\theta\tan\theta\, d\theta$	$0 \le \theta < \frac{\pi}{2}$ if $\frac{x}{a} \ge 1$ $\frac{\pi}{2} < \theta \le \pi$ if $\frac{x}{a} \le -1$	$x^2 - a^2 = a^2\sec^2\theta - a^2$ $=a^2(\sec^2\theta - 1)$ $=a^2\tan^2\theta$	

Examples: Evaluate the following:

1) $\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{dx}{\sqrt{2^2+x^2}}$



Sol.: Let $x = 2\tan\theta \;\rightarrow\; dx = 2\sec^2\theta\, d\theta$

And $4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$

$$= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln|\sec\theta + \tan\theta| + c$$

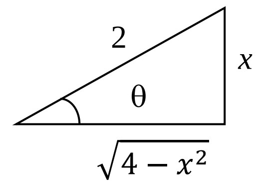
$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C = \ln\left|\left(\sqrt{4+x^2} + x\right)\right| - \ln 2 + C$$

$$= \ln\left|\left(\sqrt{4+x^2} + x\right)\right| + C_1 \qquad \text{where} \qquad C_1 = C - \ln x$$

2) $\int_{1}^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$

Sol.: Let $x = 2\sin\theta \quad \rightarrow dx = 2\cos\theta\, d\theta$

and $4 - x^2 = 4 - 4\sin^2\theta = 4(1-\sin^2\theta) = 4\cos^2\theta$



so, $\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{4\, sin^2\,\theta * \sqrt{4\cos^2\theta}} = \frac{1}{4}\int \frac{d\theta}{sin^2\,\theta}$

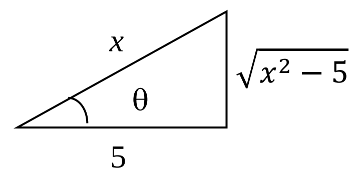
$$= \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + c = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

and

$$\int_{1}^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{-1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = -\frac{1}{4}\left[\frac{\sqrt{4-2}}{\sqrt{2}} - \frac{\sqrt{4-1}}{1}\right] = \frac{\sqrt{3}-1}{4}$$

3) $\int \frac{dx}{\sqrt{x^2-25}}$

Sol.: Let $x = 5\sec\theta \quad \rightarrow dx = 5\sec\theta\tan\theta\ d\theta$



and $x^2 - 25 = 25\sec^2\theta - 25 = 25(\sec^2\theta - 1) = 25\tan^2\theta$

$$= \int \frac{5\sec\theta\tan\theta\, d\theta}{\sqrt{25\tan^2\theta}} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + C$$

Integrals involving $ax^2 + bx + c = 0\ \ , a \neq 0$

We handle these first by completing the squares

4) $\int \frac{dx}{\sqrt{2x - x^2}}$

Sol.:

$2x - x^2 = -(x^2 - 2x + 1) + 1 = 1 - (x-1)^2$

Then substitute $u = (x-1) = du = dx$

$$So,\ \int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(x-1) + C$$

5) $\int \frac{dx}{\sqrt{x^2+2x}} = \int \frac{dx}{\sqrt{x^2+2x+1-1}} = \int \frac{dx}{\sqrt{(x+1)^2-1}}$

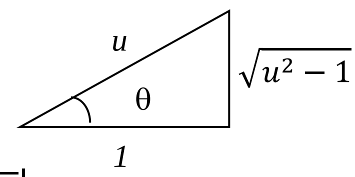
Sol.: Let $u = (x+1) \rightarrow du = dx$

$= \int \frac{du}{\sqrt{u^2-1}}$

Let $u = \sec\theta \rightarrow du = \sec\theta \tan\theta \, d\theta$

and $u^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$

$= \int \frac{\sec\theta \tan\theta \, d\theta}{\tan\theta} = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$



$= \ln\left|\frac{u}{1} + \frac{\sqrt{u^2-1}}{1}\right| + c = \ln\left|(x+1) + \sqrt{(x+1)^2-1}\right| + C$

$= \ln\left|(x+1) + \sqrt{x^2+2x+1-1}\right| + C = \ln\left|(x+1) + \sqrt{x^2+2x}\right| + C$

6) $\int \frac{xdx}{x^2-4x+8} = \int \frac{xdx}{x^2-4x+4+4} = \int \frac{xdx}{(x-2)^2+4}$

Let $u = x - 2 \rightarrow du = dx$ and $x = u + 2$

$= \int \frac{(u+2)du}{u^2+4} = \int \frac{u\,du}{u^2+4} + \int \frac{2\,du}{u^2+4}$

$= \frac{1}{2}\ln|u^2+4| + 2\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{u}{2}\right) + C$

$= \frac{1}{2}\ln|(x-2)^2+4| + 2\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{x-2}{2}\right) + C$

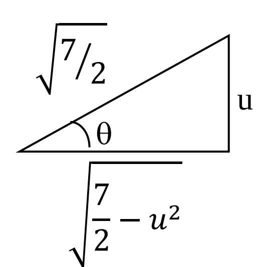
7) $\int \frac{dx}{\sqrt{5-4x-2x^2}} = \int \frac{dx}{\sqrt{5-2(x^2+2x+1-1)}}$

$= \int \frac{dx}{\sqrt{5-2(x+1)^2+2}} = \int \frac{dx}{\sqrt{7-2(x+1)^2}}$

Let $u = x + 1 \rightarrow du = dx$

$= \int \frac{du}{\sqrt{7-2u^2}} = \int \frac{du}{\sqrt{2\left(\frac{7}{2}-u^2\right)}} = \frac{1}{\sqrt{2}} \int \frac{du}{\sqrt{\frac{7}{2}-u^2}}$

$\therefore\ u = \sqrt{\frac{7}{2}}\sin\theta \rightarrow du = \sqrt{\frac{7}{2}}\cos\theta \, d\theta \rightarrow \frac{7}{2} - u^2 = \frac{7}{2} - \frac{7}{2}\sin^2\theta = \frac{7}{2}\cos^2\theta$



$$= \frac{1}{\sqrt{2}} \int \frac{\left(\sqrt{\frac{7}{2}}\right) \cos\theta \, d\theta}{\sqrt{\frac{7}{2}} \cos^2\theta} = \frac{1}{\sqrt{2}} \int d\theta = \frac{1}{\sqrt{2}} \theta + C = \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{2}{7}} u\right) + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{2}{7}} (x+1)\right) + C$$

Homework: Evaluate the following integrals;

1) $\int \sqrt{4 - x^2} \, dx$ **2)** $\int \sqrt{1 - 4x^2} \, dx$ **3)** $\int \frac{x^2 dx}{\sqrt{9 - x^2}}$ **4)** $\int \frac{dx}{x^2\sqrt{16 - x^2}}$

5) $\int \frac{dx}{(4 - x^2)^2}$ **6)** $\int \frac{x^2 dx}{\sqrt{5 + x^2}}$ **7)** $\int \frac{\sqrt{x^2 - 9} \, dx}{x}$ **8)** $\int x^3\sqrt{5 - x^2} \, dx$

9) $\int \frac{dx}{(1 - x^2)^{3/2}}$ **10)** $\int \frac{x^3 dx}{\sqrt{2 - x^2}}$ **11)** $\int e^x\sqrt{1 - e^{2x}} \, dx$ **12)** $\int \frac{\cos\theta \, d\theta}{\sqrt{2 - \sin^2\theta}}$

13) $\int x^3\sqrt{16 - x^2} \, dx$ **14)** $\int_0^{1/3} \frac{dx}{(4 - 9x^2)^2}$ **15)** $\int_0^3 \frac{x^3 dx}{(3 + x^2)^{5/2}}$ **16)** $\int \frac{dx}{1 + 2x^2 + x^4}$

17) $\int \frac{dx}{x^2 - 4x + 13}$ **18)** $\int \frac{dx}{\sqrt{2x - x^2}}$ **19)** $\int \frac{e^x dx}{\sqrt{1 + e^x + e^{2x}}}$ **20)** $\int_0^1 \sqrt{x(4 - x)} \, dx$

21) $\int (x\cos x + \sin x)\sqrt{1 + x^2\sin^2 x} \, dx$ $\quad let\ u = x\sin x$

22) $\int \sin x \cos x \sqrt{1 - \sin^4 x} \, dx$ $\quad let\ u = \sin^2 x$

syllabus

Mathematics