TECHNIQUES OF INTEGRATION

I-Basic Integration Formulas (by Substitution)

$$\int du = u + C \qquad \int a\,du = a\,u + C$$

$$\int (du + dv) = \int du + \int dv \qquad \int u^n du = \frac{u^{n+1}}{n+1} + C \quad , \quad n \neq -1$$

$$\int \cos u\,du = \sin u + C \qquad \int \sin u\,du = -\cos u + C$$

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

$$\int \frac{du}{u} = \ln|u| + C \qquad \int e^u du = e^u + C$$

$$\int a^u\,du = \frac{1}{\ln a}a^u + C \quad a > 0\ , a \neq 1$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C \qquad \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C$$

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

II-Integration by Parts

The formula for integration by parts comes from the product rule

$$\frac{d}{dx}(u\,v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

In its differential form, the rule becomes

$$d(u\,v) = u\,dv + v\,du$$

Which is written as:

$$u\,dv = d(u\,v) - v\,du$$

And integrated to give the following formula:

$$\int u\,dv = u\,v - \int v\,du$$

Examples: Evaluate the following:

1) $\int x\ \cos x\ dx$

Sol.: Let $u=x$ and $dv=\cos x\,dx$

So, $du=dx$ and $v=\sin x$

$$\therefore \int x\ \cos x\ dx = x\ \sin x - \int \sin x\ dx$$

$$= x\ \sin x + \cos x + C$$

Let's examine another choice

$u=\cos x$ $\qquad$ $dv=x\,dx$

$\therefore du=-\sin x\,dx$ $\qquad$ $v=x^2/2$

$$\int x\ \cos x\ dx = \frac{x^2}{2}\cos x - \int \frac{x^2}{2}(-\sin x)dx$$

$$= \frac{x^2}{2}\cos x + \int \frac{x^2 \sin x}{2}dx$$ this is harder than the original.

2)$\int xe^x\,dx$

Sol.: $u=x$ $\qquad$ $dv=e^x dx$

$du=dx$ $\qquad$ $v=e^x$

$$\int xe^x\,dx = xe^x - \int e^x dx = xe^x - e^x + C$$

3)$\int \ln x\,dx$

Sol.: $u = \ln x$ $\qquad$ $dv = dx$

$du = \frac{dx}{x}$ $\qquad$ $v = x$

$$\int \ln x\,dx = x\ln x - \int \frac{x\,dx}{x} = x\ln x - \int dx = x\ln x - x + C$$

4) $\int x^2\, e^{-x} dx$

Sol.: $u = x^2$ $\quad dv = e^{-x}\, dx$

$du = 2x\, dx$ $\quad v = -e^{-x}$

$\int x^2\, e^{-x} dx = -x^2\, e^{-x} - \int -e^{-x}(2x\, dx) = -x^2\, e^{-x} + 2\int xe^{-x}\, dx$

This is similar to the original except we have replaced x^2 by x

Another integration by parts applied to $\int xe^{-x}\, dx$ will complete the problem.

So, let $u = x$ $\quad dv = e^{-x}\, dx$

$du = dx$ $\quad v = -e^{-x}$

$$\int x^2\, e^{-x} dx = -x^2\, e^{-x} + 2\left[-xe^{-x} - \int (-e^{-x}) dx\right]$$

$$= -x^2\, e^{-x} - 2xe^{-x} + 2\int e^{-x}\, dx$$

$$= -x^2\, e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= -e^{-x}(x^2 + 2x + 2) + C$$

5) $\int e^x \cos x\, dx$

Sol.: Let $u = e^x$ $\quad dv = \cos x\, dx$

$du = e^x\, dx$ $\quad v = \sin x$

$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$

Again use integration by parts to integrate $\int e^x \sin x\, dx$

Let $u = e^x$ $\quad dv = \sin x\, dx$

$du = e^x\, dx$ $\quad v = -\cos x$

$$\int e^x \cos x\, dx = e^x \sin x - \left[-e^x \cos x - \int (-\cos x) e^x dx\right]$$

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x cosx\, dx$$

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x$$

$$\therefore \int e^x \cos x\, dx = \frac{e^x}{2}(\sin x + \cos x) + C$$

6) $\int sin^{-1}x dx$

Sol.: Let $u = sin^{-1}x$ $\quad dv = dx$

$$du = \frac{dx}{\sqrt{1-x^2}} \qquad v = x$$

$$\int sin^{-1}xdx = x\sin^{-1}x - \int \frac{xdx}{\sqrt{1-x^2}}$$

$$= x\sin^{-1}x - \int (1-x^2)^{(-1/2)}.x.dx$$

$$= x\sin^{-1}x - \left(\frac{-1}{2}\right)\int (1-x^2)^{(-1/2)}(-2x)dx$$

$$= x\sin^{-1}x + \frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

7) $\int xtan^{-1}xdx$

Sol.: Let $u = tan^{-1}x \qquad dv = xdx$

$$du = \frac{dx}{1+x^2} \qquad v = \frac{x^2}{2}$$

$$\int xtan^{-1}xdx = \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2/2}{1+x^2}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[\int 1 - \frac{1}{1+x^2}\right]dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}[x - \tan^{-1}x] + C$$

$$x^2+1\overline{)\,x^2}\;\;1$$
$$\underline{\mp x^2 \mp 1}$$
$$-1$$

Tabular integration:

Examples: Evaluate the following by tabular integration:

1) $\int x^2e^x\ dx$

Sol.: $u = x^2 \qquad dv = e^x\,dx$

By tabular

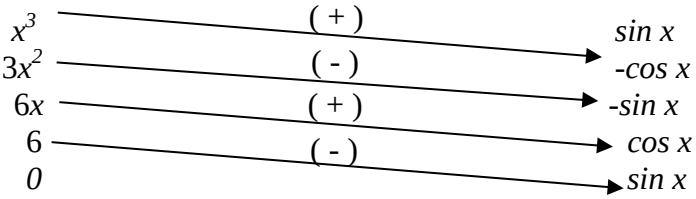
***u* and its derivatives**		***dv* and its integrals**
x^2		e^x
	(+)	
$2x$		e^x
	(-)	
	(+)	

2	e^x
0	e^x

$$\therefore \int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C$$

2) $\int x^3 \sin x \, dx$

Sol.: ***u* and its derivatives** ***dv* and its integrals**



$$\therefore \int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Homework:

a) Evaluate the following integrals

1) $\int xe^{3x} dx$ 2) $\int x^2 \sin x \, dx$ 3) $\int x \sin 2x \, dx$

4) $\int \sqrt{x} \ln x \, dx$ 5) $\int x \ln x \, dx$ 6) $\int (\ln x)^2 \, dx$

7) $\int \frac{\ln x}{\sqrt{x}} dx$ 8) $\int \ln(x^2 + 4) dx$ 9) $\int \sin^{-1} x \, dx$

10) $\int \tan^{-1}(2x) dx$ 11) $\int x \tan^{-1} x dx$ 12) $\int e^x \sin x \, dx$

13) $\int e^{2x} \cos 3x \, dx$ 14) $\int \sin(\ln x) dx$ 15) $\int \cos(\ln x) \, dx$

16) $\int x(\sec x)^2 \, dx$ 17) $\int x(\tan x)^2 dx$ 18) $\int x^3 e^{x^2} dx$

19) $\int_2^4 \sec^{-1} \sqrt{\theta} \, d\theta$ 20) $\int_1^e x^2 \ln x \, dx$

b) Evaluate the following integrals using tabular integration by parts

1) $\int (3x^2 - x + 2)e^x \, dx$ 2) $\int (x^2 + x + 1) \sin x \, dx$

3) $\int 8x^4 \cos 2x \, dx$ 4) $\int x^3 \sqrt{2x + 1} \, dx$