Inverse of Hyperbolic Functions:

All hyperbolic functions have inverses, they are:

	Inverse of hyperbolic functions	Their domains
1	$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$	$(-\infty, \infty)$
2	$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$	$[1, \infty)$
3	$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1,1)$
4	$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, \infty) \setminus [-1,1]$
5	$\operatorname{sec}h^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$(0,1]$
6	$\operatorname{csc}h^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, \infty) \setminus \{0\}$

Example: Prove that: $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$.

Sol.: let $y = \sinh^{-1} x \;\Rightarrow\; x = \sinh y \;\Rightarrow\; x = \frac{e^y - e^{-y}}{2}$

$\Rightarrow \quad e^y - 2x - e^{-y} = 0$ multiply both sides by e^y

$\Rightarrow \quad e^{2y} - 2xe^y - 1 = 0 \Rightarrow \quad e^y = \frac{2x \mp \sqrt{4x^2+4}}{2} = x \mp \sqrt{x^2+1}$

Since e^y is never negative, we must discard the minus sign.

$\therefore y = \ln(x + \sqrt{x^2+1})$

That is $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

Derivatives and Integrals:

a. Derivatives

If u is any function of x, then

1. $\frac{d}{dx}\sinh^{-1}u = \frac{du/dx}{\sqrt{1+u^2}}$
2. $\frac{d}{dx}\cosh^{-1}u = \frac{du/dx}{\sqrt{u^2-1}}$ $\quad u>1$
3. $\frac{d}{dx}\tanh^{-1}u = \frac{du/dx}{1-u^2}$ $\quad |u|<1$
4. $\frac{d}{dx}\coth^{-1}u = \frac{du/dx}{1-u^2}$ $\quad |u|>1$
5. $\frac{d}{dx}\operatorname{sech}^{-1}u = \frac{-du/dx}{u\sqrt{1-u^2}}$ $\quad 0<u<1$
6. $\frac{d}{dx}\operatorname{csch}^{-1}u = \frac{-du/dx}{u\sqrt{1+u^2}}$ $\quad u\neq 0$

b. Integrals:

If u is any function of x, then

1. $\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1}u + C$
2. $\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1}u + C$
3. $\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1}x + C & \text{if } |u|<1 \\ \coth^{-1}x + C & \text{if } |u|>1 \end{cases}$
4. $\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C$
5. $\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C$

Useful identities

1. $\operatorname{sech}^{-1}x = \cosh^{-1}\left(\frac{1}{x}\right)$
2. $\operatorname{csch}^{-1}x = \sinh^{-1}\left(\frac{1}{x}\right)$
3. $\coth^{-1}x = \tanh^{-1}\left(\frac{1}{x}\right)$

Example: Show that $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$.

Sol.: Let $y = \cosh^{-1}x \;\Rightarrow\; \therefore x = \cosh y$

And by implicit differentiation:

$$1 = \sinh y.\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}} \text{ o.k.}$$

Examples: Find dy/dx of the following functions:

1. $y = \sinh^{-1}3x$

Sol.: $\frac{dy}{dx} = \frac{3}{\sqrt{1+(3x)^2}} = \frac{3}{\sqrt{1+9x^2}}$

2. $y = \cosh^{-1}e^x$

Sol.: $\frac{dy}{dx} = \frac{e^x}{\sqrt{e^{2x}-1}}$

3. $y = 2\tanh^{-1}\left(\tan\frac{x}{2}\right)$

Sol.: $$\frac{dy}{dx} = \frac{2\sec^2\left(\frac{x}{2}\right)*\frac{1}{2}}{1-\tan^2\left(\frac{x}{2}\right)} = \frac{\frac{1}{\cos^2\left(\frac{x}{2}\right)}}{1-\frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}} = \frac{\frac{1}{\cos^2\left(\frac{x}{2}\right)}}{\frac{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}} = \frac{1}{\cos\left(2.\frac{x}{2}\right)} = \frac{1}{\cos x} = \sec x$$

4. $y = \coth^{-1}\left(\frac{1}{x}\right)$

Sol.: $\frac{dy}{dx}=\frac{-1/x^2}{1-1/x^2}=\frac{-1/x^2}{\frac{x^2-1}{x^2}}=\frac{-1}{x^2-1}=\frac{1}{1-x^2}$

5. $y=\operatorname{sec}h^{-1}(\cos x)$

Sol.: $\frac{dy}{dx}=\frac{-(-\sin x)}{\cos x\sqrt{1-\cos^2 x}}=\frac{\sin x}{\cos x \sin x}=\frac{1}{\cos x}=\sec x$

Examples: Evaluate the following integrals:

1. $\int_0^1 \frac{2.dx}{\sqrt{1+4x^2}}=\int_0^1 \frac{2.dx}{\sqrt{1+(2x)^2}}$

Let $u=2x \;\Rightarrow\; du=2.dx$, at $x=0 \;\Rightarrow\; u=0$

at $x=1 \;\Rightarrow\; u=2$

$$\int_0^2 \frac{du}{\sqrt{1+u^2}}=\left[\sinh^{-1} u\right]_0^2=\left[\sinh^{-1} 2-\sinh^{-1} 0\right]=1.4436$$

2. $\int \frac{dx}{9x^2-25}=\int \frac{dx}{25\left(\frac{9}{25}x^2-1\right)}=\frac{1}{25}\int \frac{dx}{\left(\frac{3x}{5}\right)^2-1}$

Let $u=\frac{3x}{5} \;\Rightarrow\; du=\frac{3.dx}{5} \;\Rightarrow\; dx=\frac{5.du}{3}$

$$\frac{1}{25}\int \frac{5.du/3}{u^2-1}=\frac{1}{15}\int \frac{du}{u^2-1}=\frac{-1}{15}\int \frac{du}{1-u^2}=\begin{cases}\frac{-1}{15}\tanh^{-1} u+C=\frac{-1}{15}\tanh^{-1}\left(\frac{3x}{5}\right)+C\\ \frac{-1}{15}\coth^{-1} u+C=\frac{-1}{15}\coth^{-1}\left(\frac{3x}{5}\right)+C\end{cases}$$

3. $\int \frac{dx}{\sqrt{4x^2-9}}=\int \frac{dx}{\sqrt{9\left(\frac{4}{9}x^2-1\right)}}=\int \frac{dx}{\sqrt{9\left[\left(\frac{2x}{3}\right)^2-1\right]}}$

$$=\frac{1}{3}\int \frac{\frac{2}{3}dx}{\sqrt{\left(\frac{2x}{3}\right)^2-1}}*\frac{3}{2}=\frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right)+C$$

4. $\int \frac{\sin x.dx}{\sqrt{1+\cos^2 x}}$

Let $u=\cos x \quad \Rightarrow \quad du=-\sin x.dx$

$\Rightarrow \quad \sin x.dx=-du$

$$\int \frac{-du}{\sqrt{1+u^2}}=-\sinh^{-1} u+C=-\sinh^{-1}(\cos x)+C$$

5. $\int \frac{4\tanh^{-1} x}{1-x^2}.dx$

Let $u=\tanh^{-1} x \quad \Rightarrow \quad du=\frac{dx}{1-x^2}$

$$\int 4u.du=\frac{4u^2}{2}+C=2u^2+C=2(\tanh^{-1} x)^2+C$$

6. $\int \frac{e^{\coth^{-1} x}}{1-x^2}.dx$

Let $u=\coth^{-1} x \quad \Rightarrow \quad du=\frac{dx}{1-x^2}$

$$\int e^u.du=e^u+C=e^{\coth^{-1} x}+C$$

7. . $\int \frac{\cos x.dx}{\sin x\sqrt{1+\sin^2 x}}$

Let $u=\sin x \quad \Rightarrow \quad du=\cos x.dx$

$$\int \frac{du}{u\sqrt{1+u^2}}=-\operatorname{csc}h^{-1}|u|+C=-\operatorname{csc}h^{-1}|\sin x|+C$$

Homework:

1. Find dy/dx of the following:

a. $y=\sinh^{-1} 5x$ b. $y=\sinh^{-1} e^x$ c. $y=\cosh^{-1}\sqrt{x}$

d. $y=\tanh^{-1}(x^2-1)$ e. $y=\tanh^{-1}\sin 3x$ f. $y=x\sinh^{-1}\left(\frac{1}{x}\right)$

g. $y=\frac{1}{\sinh^{-1} x^2}$ h. $y=\ln\cosh^{-1} 4x$ i. $y=\cosh^{-1}(\ln 4x)$

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2. Evaluate the following integrals:

a. $\int \frac{dx}{\sqrt{81+16x^2}}$

b. $\int \frac{dx}{\sqrt{16x^2-9}}$

c. $\int \frac{dx}{49-4x^2}$

d. $\int \frac{e^x.dx}{\sqrt{e^{2x}-16}}$

e. $\int \frac{2dx}{5-3x^2}$

f. $\int \frac{dx}{x\sqrt{9-x^4}}$

g. $\int \frac{dx}{\sqrt{5-e^{2x}}}$

h. $\int \frac{x.dx}{\sqrt{25x^2+36}}$

i. $\int \frac{dx}{\sqrt{25x^2+36}}$

j. $\int \frac{dx}{\sqrt{9-4x^2}}$

k. $\int \frac{x.dx}{\sqrt{9-4x^2}}$

l. $\int \frac{dx}{x\sqrt{9-4x^2}}$

m. $\int \frac{dx}{x\sqrt{4x^2-9}}$