syllabus

Hyperbolic Functions:

Definition and Identities:

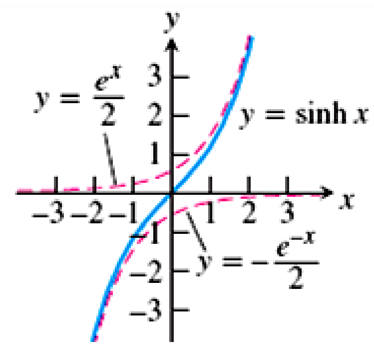
The hyperbolic cosine and hyperbolic sine functions are defined by the following equations:

Hyperbolic cosine of x: $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Note: when $x \to \infty \Rightarrow e^{-x} \to 0$ So $\cosh x \cong \dfrac{e^x}{2}$

when $x \to -\infty \Rightarrow e^{x} \to 0$ So $\cosh x \cong \dfrac{e^{-x}}{2}$

So $D_f = (-\infty, \infty)$ and $R_f = [1, \infty)$



Hyperbolic sine of x: $\sinh x = \dfrac{e^x - e^{-x}}{2}$

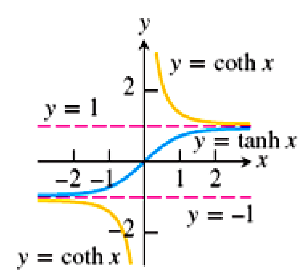
Note: when $x \to \infty \Rightarrow e^{-x} \to 0$ So $\sinh x \cong \dfrac{e^x}{2}$

when $x \to -\infty \Rightarrow e^{x} \to 0$ So $\sinh x \cong -\dfrac{e^{-x}}{2}$

So $D_f = (-\infty, \infty)$ and $R_f = (-\infty, \infty)$

The notation $\cosh x$ is often read "kosh x" and sinh x is pronounced as if spelled "cinch x" or "shine x".

Four additional hyperbolic functions are defined in terms of $\cosh x$ and $\sinh x$ as shown below:



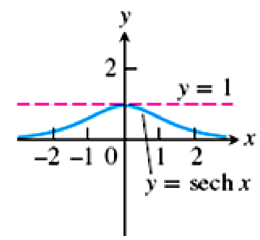
Hyperbolic tangent of x: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$D_f = (-\infty, \infty)$ and $R_f = (-1, 1)$

Hyperbolic cotangent of x: $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

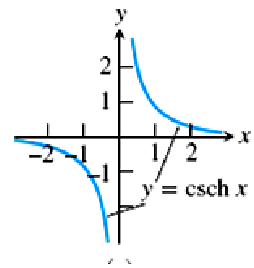
syllabus

$D_f = (-\infty, \infty) \setminus \{0\}$ and $R_f = (-\infty, -1) \cup (1, \infty)$



Hyperbolic secant of *x*: $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

$D_f = (-\infty, \infty)$ and $R_f = (0, 1]$



Hyperbolic cosecant of *x*: $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

$D_f = (-\infty, \infty) \setminus \{0\}$ and $R_f = (-\infty, \infty) \setminus \{0\}$

Identities:

	Hyperbolic functions	Trigonometric Functions
1	$\cosh^2 x - \sinh^2 x = 1$	$\cos^2 x + \sin^2 x = 1$
2	$\sinh 2x = 2\sinh x \cosh x$	$\sin 2x = 2\sin x \cos x$
3	$\cosh 2x = \cosh^2 x + \sinh^2 x$	$\cos 2x = \cos^2 x - \sin^2 x$
4	$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$	$\cos^2 x = \dfrac{1 + \cos 2x}{2}$
5	$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$	$\sin^2 x = \dfrac{1 - \cos 2x}{2}$
6	$\tanh^2 x = 1 - \operatorname{sech}^2 x$	$\tan^2 x = \sec^2 x - 1$
7	$\coth^2 x = 1 + \operatorname{csch}^2 x$	$\cot^2 x = \csc^2 x - 1$

Examples: Prove that:

1. $\cosh^2 x - \sinh^2 x = 1$

Sol.: left side: $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 + \left(\dfrac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} + \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}$$

$$= \frac{e^{2x}}{4} + \frac{2e^0}{4} + \frac{e^{-2x}}{4} - \frac{e^{2x}}{4} + \frac{2e^0}{4} - \frac{e^{-2x}}{4} = \frac{4e^0}{4}$$

$=1=$ right side o.k.

2. $\operatorname{csch}^2 x = \coth^2 x - 1$

Sol.: right side: $\coth^2 x - 1 = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - 1 = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} - 2 + e^{-2x}} - 1$

$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{e^{2x} - 2 + e^{-2x}}$$

$$\frac{4}{e^{2x} + 2 + e^{-2x}} = \left(\frac{2}{e^x + e^{-x}}\right)^2$$

$= \operatorname{csch}^2 x =$ left side o.k.

3. $\cosh^2 x = \frac{\cosh 2x + 1}{2}$

Sol.: right side: $\frac{\cosh 2x + 1}{2} = \frac{\left(e^{2x} + e^{-2x}\right)/2 + 1}{2}$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$= \left(\frac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x =$ left side o.k.

4. $\cosh x + \sinh x = e^x$

Sol.: left side: $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$

$$= \frac{1}{2}\left(e^x + e^{-x} + e^x - e^{-x}\right) = \frac{2e^x}{2}$$

$= e^x =$ right side o.k.

Examples: Solve the following equations:

1. $5\cosh x - 3\sinh x = 5$

Sol.: $5\frac{e^x + e^{-x}}{2} - 3\frac{e^x - e^{-x}}{2} = 5$

$$5e^x + 5e^{-x} - 3e^x + 3e^{-x} = 10 \Rightarrow 2e^x + 8e^{-x} = 10 \Rightarrow e^x + 4e^{-x} = 5$$

$$\Rightarrow \quad e^x + \frac{4}{e^x} = 5 \quad \Rightarrow \quad \frac{e^{2x}+4}{e^x} = 5 \quad \Rightarrow \quad e^{2x} + 4 = 5e^x \quad \Rightarrow \quad e^{2x} - 5e^x + 4 = 0$$

$$\Rightarrow \quad (e^x - 4)(e^x - 1) = 0$$

$\therefore$ either $(e^x - 4) = 0 \quad \Rightarrow \quad e^x = 4 \quad \Rightarrow \quad x = \ln 4$

or $(e^x - 1) = 0 \quad \Rightarrow \quad e^x = 1 \quad \Rightarrow \quad x = \ln 1 = 0$

2. $3\cosh x - 2\sinh x = 10$

Sol.: $3\dfrac{e^x + e^{-x}}{2} - 2\dfrac{e^x - e^{-x}}{2} = 10$

$$3e^x + 3e^{-x} - 2e^x + 2e^{-x} = 20 \quad \Rightarrow \quad e^x + 5e^{-x} = 20$$

$$\Rightarrow \quad e^{2x} + 5 = 20e^x \quad \Rightarrow \quad e^{2x} - 20e^x + 5 = 0$$

$$e^x = \frac{-(-20) \mp \sqrt{(-20)^2 - 4(1)(5)}}{2(1)} = \frac{20 \mp \sqrt{400-20}}{2} = \frac{20 \mp \sqrt{380}}{2}$$

$\therefore$ either $e^x = 19.74 \quad \Rightarrow \quad x = \ln 19.74 = 2.98$

or $e^x = 0.254 \quad \Rightarrow \quad x = \ln 0.254 = -1.373$

Derivatives of Hyperbolic Function:

If u is any function of x, then:

	Derivative of hyperbolic functions	Derivative of trigonometric functions
1	$\frac{d}{dx}\sinh u = \cosh u.\frac{du}{dx}$	$\frac{d}{dx}\sin u = \cos u.\frac{du}{dx}$
2	$\frac{d}{dx}\cosh u = \sinh u.\frac{du}{dx}$	$\frac{d}{dx}\cos u = -\sin u.\frac{du}{dx}$
3	$\frac{d}{dx}\tanh u = \sec h^2 u.\frac{du}{dx}$	$\frac{d}{dx}\tan u = \sec^2 u.\frac{du}{dx}$
4	$\frac{d}{dx}\coth u = -\csc h^2 u.\frac{du}{dx}$	$\frac{d}{dx}\cot u = -\csc^2 u.\frac{du}{dx}$
5	$\frac{d}{dx}\sec hu = -\sec hu \tanh u.\frac{du}{dx}$	$\frac{d}{dx}\sec u = \sec u \tan u.\frac{du}{dx}$

6	$\frac{d}{dx}\csc hu = -\csc hu \coth u.\frac{du}{dx}$	$\frac{d}{dx}\csc u = -\csc u \cot u.\frac{du}{dx}$

Examples: Prove that:

1. $\frac{d}{dx}\sinh x = \cosh x$

Sol.: $\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$ o.k.

2. $\frac{d}{dx}\csc hx = -\csc hx \coth x$

Sol.: $\frac{d}{dx}\csc hu = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = \frac{\sinh x * (0) - 1 * \cosh x}{\sinh^2 x}$

$= \frac{-\cosh x}{\sinh^2 x} = \frac{-1}{\sinh x}.\frac{\cosh x}{\sinh x} = -\csc hx \coth x$ o.k.

Examples: Find $\frac{dy}{dx}$ of the following:

1. $y = \sinh 3x$

Sol.: $\frac{dy}{dx} = \cosh 3x * 3 = 3\cosh 3x$

2. $y = \tanh(1 + x^3)$

Sol.: $\frac{dy}{dx} = \sec h^2(1 + x^3) * 3x^2 = 3x^2 \sec h^2(1 + x^3)$

3. $y = \coth\frac{1}{x}$

Sol.: $\frac{dy}{dx} = -\csc h^2\left(\frac{1}{x}\right) * \left(\frac{-1}{x^2}\right) = \frac{1}{x^2}\csc h^2\left(\frac{1}{x}\right)$

4. $y = x \sec hx^2$

Sol.: $\frac{dy}{dx} = x\left(-\sec hx^2 . \tanh x^2 . 2x\right) + \sec hx^2$

$= -2x^2 \sec hx^2 . \tanh x^2 + \sec hx^2$

5. $y = \csc h^2(x^2+1)$

Sol.: $\frac{dy}{dx} = 2\csc h(x^2+1)\left[-\csc h(x^2+1)\coth(x^2+1) * 2x\right]$

$= -4x\csc h^2(x^2+1)\coth(x^2+1)$

6. $y = \ln\tanh 2x$

Sol.: $\frac{dy}{dx} = \frac{\sec h^2 2x * 2}{\tanh 2x} = \frac{2\frac{1}{\cosh^2 2x}}{\frac{\sinh 2x}{\cosh 2x}} = \frac{2}{\cosh^2 2x} * \frac{\cosh 2x}{\sinh 2x}$

$= \frac{2*2}{2\cosh 2x.\sinh 2x} = \frac{4}{\sinh 4x} = 4\csc h4x$

7. $y = (\sinh x)^x$

Sol.: $\ln y = x\ln\sinh x$

$\frac{1}{y}.\frac{dy}{dx} = x.\frac{\cosh x}{\sinh x} + \ln\sinh x$

$\frac{dy}{dx} = y\left(x.\frac{\cosh x}{\sinh x} + \ln\sinh x\right) = (\sinh x)^x(x\coth x + \ln\sinh x)$

Integrals of Hyperbolic Function:

If u is any function of x, then:

1. $\int \sinh u.du = \cosh u + C$
2. $\int \cosh u.du = \sinh u + C$
3. $\int \sec h^2 u.du = \tanh u + C$
4. $\int \csc h^2 u.du = -\coth u + C$
5. $\int \sec hu.\tanh u.du = -\sec hu + C$
6. $\int \csc hu.\coth u.du = -\csc hu + C$

Examples: Evaluate the following integrals:

1. $\int \coth 5x.dx = \int \frac{\cosh 5x}{\sinh 5x}.dx$

Let $u = \sinh 5x \quad \Rightarrow \quad du = \cosh 5x.5dx$ ➔ $\therefore \cosh 5x.dx = \frac{du}{5}$

$$\int \frac{1}{u}.\frac{du}{5} = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

2. $\int_0^{\ln 2} 4e^x \sinh x.dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}.dx = \int_0^{\ln 2}\left(2e^{2x} - 2\right)dx$

$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = \left[\left(e^{2\ln 2} - 2\ln 2\right) - \left(e^0 - 2*0\right)\right]$$

$$= e^{\ln 2^2} - 2\ln 2 - 1 = 4 - 2\ln 2 - 1 = 3 - 2\ln 2$$

3. $\int_0^1 \sinh^2 x.dx = \int_0^1 \frac{\cosh 2x - 1}{2}.dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$

$$= \frac{1}{2}\left[\left(\frac{\sinh 2}{2} - 1\right) - \left(\frac{\sinh 0}{2} - 0\right)\right] = \frac{\sinh 2}{4} - \frac{1}{2} = 0.40672$$

4. $\int \tanh 3x.\sec h^2 3x.dx$

Sol.: let $u = \tanh 3x \quad \Rightarrow \quad du = 3\sec h^2 3x.dx$ ➔ $\therefore \sec h^2 3x.dx = \frac{du}{3}$

$$\int u.\frac{du}{3} = \frac{u^2}{2*3} + C = \frac{\tanh^2 3x}{6} + C$$

5. $\int e^{\coth x}.\csc h^2 x.dx$

Sol.: let $u = \coth x \quad \Rightarrow \quad du = -\csc h^2 x.dx$ ➔ $\therefore \csc h^2 x.dx = -du$

$$\int e^u(-du) = -e^u + C = -e^{\coth x} + C$$

6. $\int \frac{\csc h^2\sqrt{x}}{\sqrt{x}}.dx$

Sol.: let $u^2 = x \quad \Rightarrow \quad 2u.du = dx$

$$\int \frac{\csc h^2 u}{u}.2u.du = 2\int \csc h^2 u.du = -2\coth u + C = -2\coth\sqrt{x} + C$$

[syllabus]

Mathematics