## Inverse of Trigonometric functions:

**1.** The **arcsine of *x* ($\sin^{-1}x$)** is the angle in $[-\pi/2, \pi/2]$ whose sine is $x$.

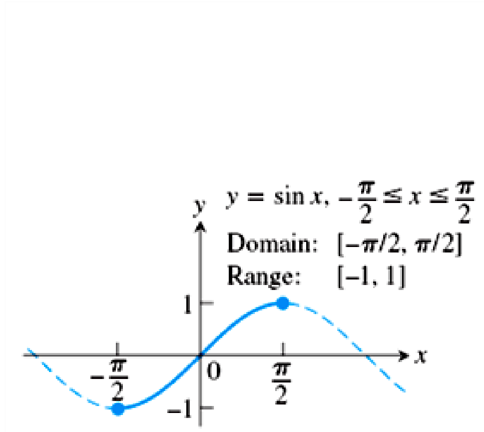
The function $y=\sin x$ is one-to-one, if we restrict its domain to the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ . It has an inverse which is denoted by:

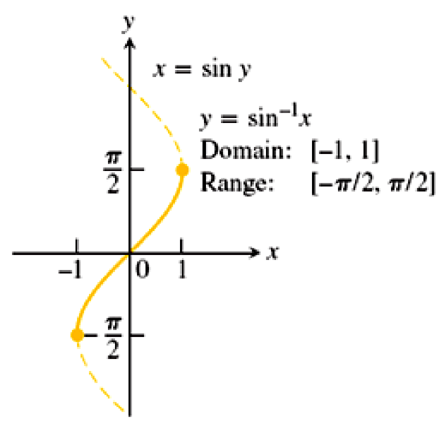
$$\boxed{y = \sin^{-1} x}$$

and is sometimes written as $y=$**arc**$\sin x$

and for the function $y=\sin^{-1}x$

$D_f = [-1,1]$ and $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$




***Note:*** The graph of $\sin^{-1}x$ is symmetric about the origin because that the graph of $\sin x$ is symmetric about the origin this means that

$$\sin^{-1}(-x) = -\sin^{-1}x$$

**2.** The **arccosine of *x* ($\cos^{-1}x$)** is the angle in $[0, \pi]$ whose cosine is $x$.

The function $y=\cos x$ is one-to-one, if we restrict its domain to the interval $0 \le x \le \pi$ . It has an inverse which is denoted by:

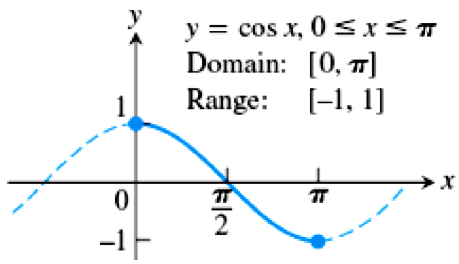
$$\boxed{y = \cos^{-1} x}$$

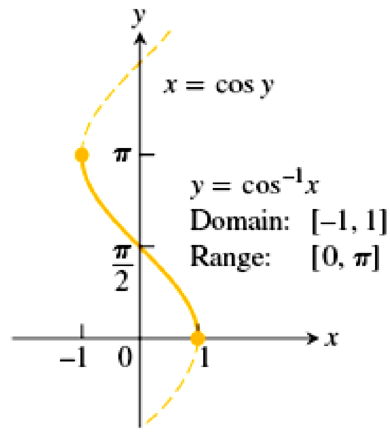
and is sometimes written as $y=$**arc**$\cos x$

and for the function $y=\cos^{-1}x$

$D_f = [-1,1]$ and $R_f = [0, \pi]$

**Note:** The graph of $y = \cos^{-1} x$ has no such symmetry




**Note:** We can see from the figures below the following identities

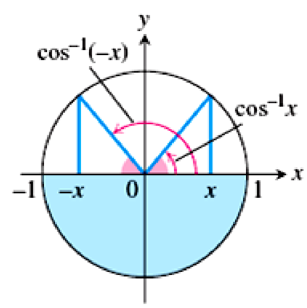
1. $\cos^{-1} x + \cos^{-1}(-x) = \pi$

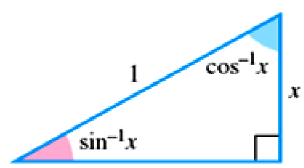
   $\therefore \cos^{-1}(-x) = \pi - \cos^{-1} x$

and form the triangle

2. $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$ for

$x > 0$




**3.** The **arctangent of *x* (tan$^{-1}$*x*)** is the angle in $(-\pi/2, \pi/2)$ whose tangent is $x$.

The function $y = \tan x$ is one-to-one, if we restrict its domain to the interval $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ . It has an inverse which is denoted by:

$$\boxed{y = \tan^{-1} x}$$

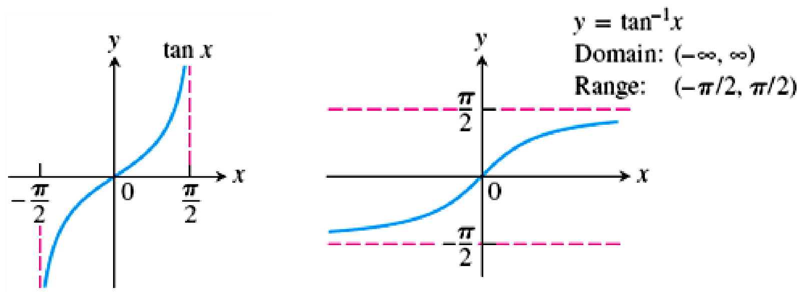
and is sometimes written as $y=$**arc**$\tan x$

and for the function $y = \tan^{-1} x$

$D_f = (-\infty, \infty)$ and $R_f = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

***Note:*** The graph of $\tan^{-1}x$ is symmetric about the origin because that the graph of $\tan x$ is symmetric about the origin, this means that

$$\tan^{-1}(-x)=-\tan^{-1}x$$



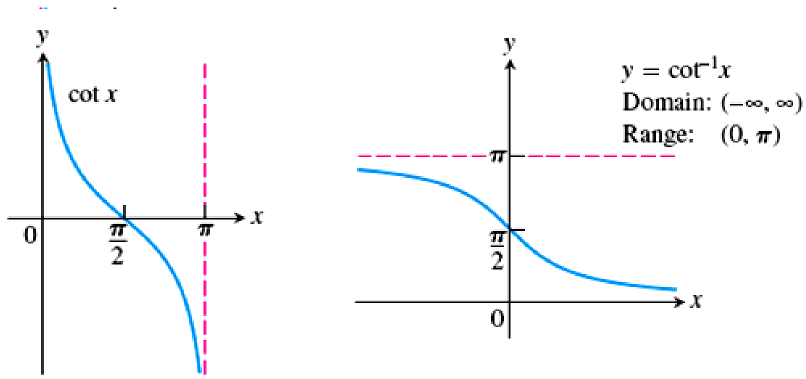

**4.** The **arctcoangent of** $x$ **($\cot^{-1}x$)** is the angle in $(0,\pi)$ whose cotangent is $x$.

The function $y=\cot x$ is one-to-one, if we restrict its domain to the interval $0<x<\pi$ . It has an inverse which is denoted by:

$$\boxed{y=\cot^{-1}x}$$

and is sometimes written as $y=$**arc**$\cot x$

and for the function $y=\cot^{-1}x$




$D_f=(-\infty,\infty)$ and $R_f=(0,\pi)$.

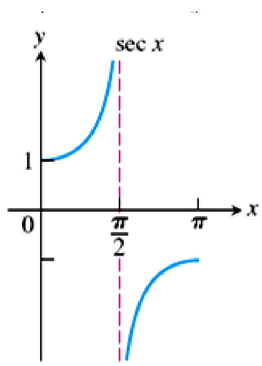
**5.** The function $y=\sec x$ is one-to-one, if we restrict its domain to the interval $\{x: 0 \le x \le \pi\} \setminus \{\frac{\pi}{2}\}$. It has an inverse which is denoted by:

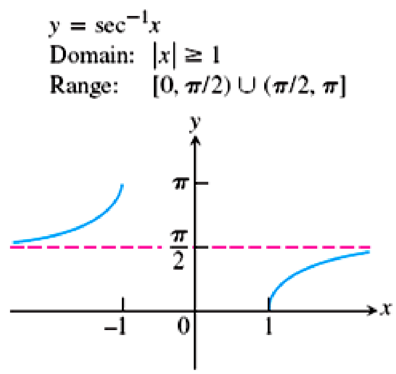
$$\boxed{y = \sec^{-1} x}$$

and is sometimes written as $y$ = **arc**$\sec x$

and for the function $y=\sec^{-1}x$

$D_f = R \setminus (-1,1)$ and $R_f = [0, \pi] \setminus \{\frac{\pi}{2}\}$.



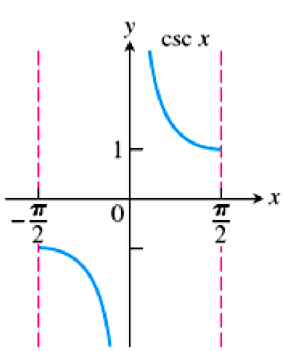

**6.** The function $y=\csc x$ is one-to-one, if we restrict its domain to the interval $\{x: -\frac{\pi}{2} \le x \le \frac{\pi}{2}\} \setminus \{0\}$. It has an inverse which is denoted by:

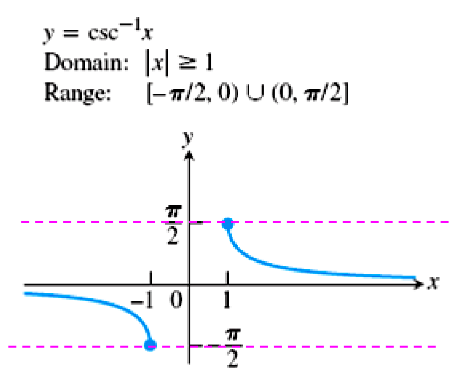
$$\boxed{y = \csc^{-1} x}$$

and is sometimes written as $y$=**arc**$\csc x$

and for the function $y=\csc^{-1}x$

$D_f = R \setminus (-1,1)$ and $R_f = [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$.

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*Note*: To find $\sec^{-1}x$, $\csc^{-1}x$ and $\cot^{-1}x$, use the following identities:

1. $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$
2. $\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$
3. $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$

***Example 1:*** Show that $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$.

**Sol.:** Let $z =$ right side $= \cos^{-1}\left(\frac{1}{x}\right) \Rightarrow \cos z = \cos\left(\cos^{-1}\frac{1}{x}\right) \Rightarrow \cos z = \frac{1}{x}$

$$\Rightarrow x = \frac{1}{\cos z} = \sec z$$

$\sec^{-1} x = \sec^{-1}(\sec z) \Rightarrow \sec^{-1} x = z =$ left side o.k.

***Example 2:*** Show that: $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$.

**Sol.:** Let $z =$ right side $= \frac{\pi}{2} - \tan^{-1} x \Rightarrow \tan^{-1} x = \frac{\pi}{2} - z$

$$\Rightarrow \tan(\tan^{-1} x) = \tan\left(\frac{\pi}{2} - z\right) \Rightarrow x = \frac{\sin\left(\frac{\pi}{2} - z\right)}{\cos\left(\frac{\pi}{2} - z\right)} = \frac{\cos z}{\sin z} = \cot z$$

$\therefore z = \cot^{-1} x =$ left side o.k.

***Examples:*** Find the limits of the following:

1. $\lim_{x\to 1^-} \sin^{-1} x = \sin^{-1} 1^- = \frac{\pi}{2}$

2. $\lim_{x\to -1^+} \cos^{-1} x = \cos^{-1}(-1^+) = \pi$

3. $\lim_{x\to \infty} \tan^{-1} x = \tan^{-1} \infty = \frac{\pi}{2}$

4. $\lim_{x\to -\infty} \tan^{-1} x = \tan^{-1}(-\infty) = -\frac{\pi}{2}$

5. $\lim_{x\to \infty} \sec^{-1} x = \lim_{x\to \infty} \cos^{-1}(\frac{1}{x}) = \cos^{-1} 0 = \frac{\pi}{2}$

6. $\lim_{x\to -\infty} \sec^{-1} x = \lim_{x\to -\infty} \cos^{-1}(\frac{1}{x}) = \cos^{-1} 0 = \frac{\pi}{2}$

7. $\lim_{x\to \infty} \csc^{-1} x = \lim_{x\to \infty} \sin^{-1}(\frac{1}{x}) = \sin^{-1} 0 = 0$

8. $\lim_{x\to -\infty} \csc^{-1} x = \lim_{x\to -\infty} \sin^{-1}(\frac{1}{x}) = \sin^{-1} 0 = 0$

**The Derivative of Inverse Trigonometric Functions:**

***Example 1:*** If $y=\sin^{-1}x$, then find $dy/dx$.

**Sol.:** $y=\sin^{-1}x \Rightarrow x=\sin y$

$$1 = \cos y.\frac{dy}{dx} \quad \text{(using implicit differentiation)}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \text{ (remember that } \cos y>0 \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2})$$

$$\therefore \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

***Example 2:*** If $y=\sec^{-1}x$, then find $dy/dx$.

**Sol.:** $y=\sec^{-1}x \Rightarrow x=\sec y$

$$1 = \sec y \tan y . \frac{dy}{dx}$$ (using implicit differentiation)

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\left(\mp\sqrt{\sec^2 y - 1}\right)} = \frac{1}{x\left(\mp\sqrt{x^2-1}\right)}$$

(remember that $-\infty < \tan y < \infty$ for $\{x:\ 0 < x < \pi\}\backslash\{\frac{\pi}{2}\}$

$$\therefore \frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$

***Example 3:*** If $y=\tan^{-1}x$, then find $dy/dx$.

**Sol.:** $y=\tan^{-1}x \Rightarrow x=\tan y$

$$1 = \sec^2 y . \frac{dy}{dx}$$ (using implicit differentiation)

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2+1}$$

$$\therefore \frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}$$

**In general:** If $u$ is a function of $x$:

1. $\frac{d}{dx}\sin^{-1} u = \frac{du/dx}{\sqrt{1-u^2}} \quad |u| < 1$
2. $\frac{d}{dx}\cos^{-1} u = \frac{-du/dx}{\sqrt{1-u^2}} \quad |u| < 1$
3. $\frac{d}{dx}\tan^{-1} u = \frac{du/dx}{1+u^2}$
4. $\frac{d}{dx}\cot^{-1} u = \frac{-du/dx}{1+u^2}$
5. $\frac{d}{dx}\sec^{-1} u = \frac{du/dx}{|u|\sqrt{u^2-1}} \quad |u| > 1$
6. $\frac{d}{dx}\csc^{-1} u = \frac{-du/dx}{|u|\sqrt{u^2-1}} \quad |u| > 1$

***Examples***: Find $dy/dx$ of the following functions:

1. $y = \sin^{-1} x^2$

**Sol.:** $\dfrac{dy}{dx} = \dfrac{2x}{\sqrt{1-(x^2)^2}} = \dfrac{2x}{\sqrt{1-x^4}}$

2. $y = \tan^{-1} \sqrt{x+1}$

**Sol.:** $\dfrac{dy}{dx} = \dfrac{\frac{1}{2\sqrt{x+1}}}{1+\left(\sqrt{x+1}\right)^2} = \dfrac{1}{2\sqrt{x+1}} * \dfrac{1}{1+x+1} = \dfrac{1}{2\sqrt{x+1}} * \dfrac{1}{2+x}$

3. $y = \sec^{-1} 3x$

**Sol.:** $\dfrac{dy}{dx} = \dfrac{3}{|3x|\sqrt{(3x)^2-1}} = \dfrac{1}{|x|\sqrt{9x^2-1}}$

4. $y = x \sin^{-1} 3x$

**Sol.:** $\dfrac{dy}{dx} = x * \dfrac{3}{\sqrt{1-(3x)^2}} + \sin^{-1} 3x * 1 = \dfrac{3x}{\sqrt{1-9x^2}} + \sin^{-1} 3x$

5. $y^2 \sin x + y = \arctan y$

**Sol.:** $2y.y`\sin x + y^2 \cos x + y` = \dfrac{y`}{1+y^2}$

$$y`(2y\sin x + 1 - \frac{1}{1+y^2}) = -y^2 \cos x$$

$$y`(\frac{2y\sin x(1+y^2)+(1+y^2)-1}{1+y^2}) = -y^2 \cos x$$

$$y`(\frac{2y\sin x + 2y^3 \sin x + 1 + y^2 - 1}{1+y^2}) = -y^2 \cos x$$

$$y` = \frac{-y^2 \cos x(1+y^2)}{2y\sin x + 2y^3 \sin x + y^2} = \frac{-y\cos x(1+y^2)}{2\sin x + 2y^2 \sin x + y}$$