Other Exponential and Logarithmic Function

a. The Function a^x

If a is a positive number and x is any number, we define the function a^x "a to the x" by the equation:

$$a^x = e^{x\ln a}$$

Assume $y = a^x$

$\therefore \ln y = \ln a^x$ "take the logarithm of both sides"

$\ln y = x\ln a$

$e^{\ln y} = e^{x\ln a}$ "exponentiate of both sides"

$\therefore y = e^{x\ln a}$

Laws of Exponents

If $a > 0$ any x and y then:

1. $a^x . a^y = a^{x+y}$
2. $a^{-x} = \dfrac{1}{a^x}$
3. $\dfrac{a^x}{a^y} = a^{x-y}$
4. $(a^x)^y = (a^y)^x = a^{xy}$

The Derivative of a^x:

Let $y = a^x = e^{x\ln a}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = e^{x\ln a} * \ln a = a^x * \ln a$$

$$\therefore \frac{d}{dx} a^x = \ln a * a^x$$

In general:

$$\frac{d}{dx} a^u = \ln a * a^u * \frac{du}{dx}$$

where u is a differential function of x.

Examples: Find dy/dx of the following:

1. $y = 3^{-x}$

Sol.: $\dfrac{dy}{dx} = 3^{-x} * \ln 3(-1) = -\ln 3 * 3^{-x}$

2. $y = 3^{\sin x}$

Sol.: $\dfrac{dy}{dx} = 3^{\sin x} * \ln 3 * (\cos x)$

Another Solution:

$y = 3^{\sin x}$

$\ln y = \ln 3^{\sin x}$ "take the logarithm of both sides"

$\ln y = \sin x * \ln 3$

$\dfrac{1}{y} . \dfrac{dy}{dx} = \cos x * \ln 3$

$\therefore \dfrac{dy}{dx} = y * \cos x * \ln 3$

$= 3^{\sin x} * \cos x * \ln 3$

Other Power Functions:

Examples: Find *dy/dx* of the following:

1. $y = x^x$, $x > 0$

Sol.: Take the logarithm of both sides

$\ln y = \ln x^x$

$\ln y = x \ln x$

Differentiate using implicit differentiation

$\dfrac{1}{y} . \dfrac{dy}{dx} = x * \dfrac{1}{x} + \ln x$

$\therefore \dfrac{dy}{dx} = y\left[\cancel{x} * \dfrac{1}{\cancel{x}} + \ln x\right] = x^x [1 + \ln x]$

2. $y = x^{\ln x}$, $x > 0$

Sol.: $\ln y = (\ln x)^{\ln x} \Rightarrow \ln y = \ln x * \ln x \Rightarrow \ln y = \ln^2 x$

$$\frac{1}{y}.\frac{dy}{dx} = 2\ln x * \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{2\ln x}{x}\right] = 2x^{\ln x}\left[\frac{\ln x}{x}\right]$$

3. $y = (\sin x)^{\tan x}$

Sol.: $\ln y = \tan x \ln(\sin x)$

$$\frac{1}{y}.\frac{dy}{dx} = \tan x * \frac{\cos x}{\sin x} + \sec^2 x \ln \sin x$$

$$\therefore \frac{dy}{dx} = y\left[1 + \sec^2 x \ln \sin x\right] = \sin x^{\tan x}\left[1 + \sec^2 x \ln \sin x\right]$$

4. $y = (\sqrt{x})^x$

Sol.: $\ln y = x \ln \sqrt{x} \quad \Rightarrow \quad \ln y = x \ln x^{1/2} \quad \Rightarrow \quad \ln y = \frac{1}{2} x \ln x$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{2}\left[x.\frac{1}{x} + \ln x\right]$$

$$\therefore \frac{dy}{dx} = \frac{y}{2}\left[1 + \ln x\right] = \frac{(\sqrt{x})^x}{2}\left[1 + \ln x\right]$$

5. $y = x^{(x+1)}$

Sol.: $\ln y = (x+1)\ln x$

$$\frac{1}{y}.\frac{dy}{dx} = (x+1).\frac{1}{x} + \ln x$$

$$\frac{dy}{dx} = y\left[\frac{x+1}{x} + \ln x\right] = x^{(x+1)}\left[\frac{x+1}{x} + \ln x\right]$$

The Integral of a^x:

If $a \neq 1$ so that $\ln a \neq 0$, you know that:

$$\frac{d}{dx} a^u = a^u * \ln a * \frac{du}{dx}$$

$$\therefore \frac{1}{\ln a}\left[\frac{d}{dx} a^u\right] = a^u \frac{du}{dx}$$ "divide both sides by $\ln a$"

Integrate of both sides:-

$$\int\left[a^{u}\frac{du}{dx}\right]dx=\int\frac{1}{\ln a}\left[\frac{d}{dx}a^{u}\right]dx=\frac{1}{\ln a}\int\left[\frac{d}{dx}a^{u}\right]dx$$

$$\boxed{\int a^{u}du=\frac{1}{\ln a}a^{u}+C}$$

Examples: Evaluate the following integrals:

1. $\int 2^{x}.dx=\frac{1}{\ln 2}.2^{x}+C$

Another solution:

$$\int 2^{x}.dx=\int e^{x\ln 2}.dx=\frac{1}{\ln 2}\int e^{x\ln 2}.\ln 2.dx=\frac{1}{\ln 2}e^{x\ln 2}+C=\frac{1}{\ln 2}2^{x}+C$$

2. $\int 2^{\sin x}\cos x.dx$

Sol.: let $u=\sin x \Rightarrow du=\cos x.dx$

$$\int 2^{u}.du=\frac{1}{\ln 2}.2^{u}+C=\frac{1}{\ln 2}.2^{\sin x}+C$$

3. $\int 3^{\tan 7x}\sec^{2}7x.dx$

Sol.: let $u=\tan 7x \Rightarrow du=\sec^{2}7x*7dx$

$$\therefore \sec^{2}7x.dx=\frac{du}{7}$$

$$\int 3^{u}.\frac{du}{7}=\frac{1}{7}*\frac{1}{\ln 3}*3^{u}+C=\frac{1}{7}*\frac{1}{\ln 3}*3^{\tan 7x}+C=\frac{1}{7\ln 3}*3^{\tan 7x}+C$$

4. $\int 2^{x}\cos 2^{x}.dx$

Sol.: let $u=2^{x} \Rightarrow du=\ln 2*2^{x}dx$

$$\therefore 2^{x}.dx=\frac{du}{\ln 2}$$

$$\int \cos u.\frac{du}{\ln 2} = \frac{\sin u}{\ln 2} + C = \frac{\sin(2^x)}{\ln 2} + C$$

5. $\int \frac{\sec^3 x + e^{\sin x}}{\sec x}.dx = \int \left[\frac{\sec^3 x}{\sec x} + \frac{e^{\sin x}}{\sec x}\right].dx = \int \left[\sec^2 x + e^{\sin x}.\cos x\right].dx = \tan x + e^{\sin x} + C$

6. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}.dx$

Sol.: let $u = e^{x} + e^{-x} \Rightarrow du = (e^{x} - e^{-x}).\, dx$

$$\int \frac{du}{u} = \ln|u| + C = \ln|e^x + e^{-x}| + C$$

7. $\int e^{2t}\sqrt{1+e^{2t}}.dt = \int e^{2t}(1+e^{2t})^{1/2}.dt$

Sol.: let $u = 1 + e^{2t} \Rightarrow du = 2e^{2t}.\, dt$

$$\therefore e^{2t}.dt = \frac{du}{2}$$

$$\int u^{1/2}.\frac{du}{2} = \frac{1}{2}.\frac{u^{3/2}}{3/2} + C = \frac{1}{2}.\frac{2u^{3/2}}{3} + C = \frac{1}{3}(1+e^{2t})^{3/2} + C$$

8. $\int x4^{-x^2}.dx$

Sol.: let $u = 4^{-x^2} \Rightarrow du = \ln 4 * 4^{-x^2} * (-2x).dx$

$$\therefore x4^{-x^2}.dx = -\frac{du}{2\ln 4}$$

$$\int -\frac{du}{2\ln 4} = \frac{-u}{2\ln 4} + C = \frac{-4^{-x^2}}{2\ln 4} + C$$

b. Base *a* Logarithms:-

$\log_a x$ = inverse of a^x where $a > 0$ and $a \neq 1$

$\therefore \log_a a^x = x$ for all x

and $a^{\log_a x} = x$ for all $x > 0$

Evaluation of $\log_a x$:

Use $a^{\log_a x} = x$

So $a^{\log_a x} = x$ "take the logarithm of both sides"

$$\ln a^{\log_a x} = \ln x$$

$$\log_a x * \ln a = \ln x$$

$$\therefore \boxed{\log_a x = \frac{\ln x}{\ln a}}$$

Rules of arithmetic for base *a* logarithms:

For any positive numbers u and v and for any exponent n,

1. $\log_a u.v = \log_a u + \log_a v$
2. $\log_a \frac{u}{v} = \log_a u - \log_a v$
3. $\log_a u^n = n \log_a u$
4. $\log_a \sqrt[n]{u} = \frac{1}{n} \log_a u$

Derivatives and integrals involving $\log_a x$:

If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}.\frac{d}{dx}(\ln u) = \frac{1}{\ln a}.\frac{1}{u}\frac{du}{dx}$$

$$\boxed{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}.\frac{1}{u}\frac{du}{dx}}$$

Examples 1: Find dy/dx of the following functions:

a. $y = \log_{10}(3x+1)$

Sol.: $y = \log_{10}(3x+1) = \frac{\ln(3x+1)}{\ln 10}$

$$\therefore \frac{dy}{dx} = \frac{1}{\ln 10}.\frac{3}{3x+1}$$

b. $y = \log_5 x^2$

Sol.: $y=\log_5 x^2=\frac{\ln x^2}{\ln 5}=\frac{2\ln x}{\ln 5}$

$$\therefore \frac{dy}{dx}=\frac{2}{\ln 5}.\left(\frac{1}{x}\right)=\frac{2}{x\ln 5}$$

c. $y=\log_{10} e^x$

Sol.: $y=\log_{10} e^x=\frac{\ln e^x}{\ln 10}=\frac{x}{\ln 10}$

$$\therefore \frac{dy}{dx}=\frac{1}{\ln 10}$$

d. $y=\log_5(x^2+1)^2$

Sol.: $y=\log_5(x^2+1)^2=2\log_5(x^2+1)=\frac{2\ln(x^2+1)}{\ln 5}$

$$\therefore \frac{dy}{dx}=\frac{2}{\ln 5}.\frac{2x}{x^2+1}=\frac{4x}{\ln 5(x^2+1)}$$

Examples 2: Evaluate the following integrals:

a. $\int\frac{\log_2 x}{x}.dx$

Sol.: $\int\frac{\log_2 x}{x}.dx=\int\frac{\ln x}{\ln 2}.\frac{dx}{x}=\frac{1}{\ln 2}\int\ln x.\frac{dx}{x}=\frac{1}{\ln 2}\frac{(\ln x)^2}{2}+C=\frac{\ln^2 x}{2\ln 2}+C$

b. $\int\frac{\log_{10}(x+2)}{x+2}.dx$

Sol.: $\int\frac{\log_{10}(x+2)}{x+2}.dx=\int\frac{\ln(x+2)}{\ln 10}.\frac{dx}{(x+2)}$

Let $u=\ln(x+2) \Rightarrow du=\frac{dx}{(x+2)}$

$$\frac{1}{\ln 10}\int u.du=\frac{1}{\ln 10}.\frac{u^2}{2}+C=\frac{\ln^2(x+2)}{2\ln 10}+C$$

The Graph of a^x

If $y=a^x$ and both x and a are positive and $a\neq 1$, then

- The domain and range of $y = a^x$ is the same as the domain and range of the function $y = e^x$.

  $\therefore D_f=(-\infty, \infty)$ and $R_f=(0, \infty)$

- First derivative test

  $$\frac{d}{dx}a^x = \ln a * a^x$$

  So if $a > 1 \Rightarrow \ln a * a^x > 0 \Rightarrow a^x$ is an increasing function

  if $a < 1 \Rightarrow \ln a * a^x < 0 \Rightarrow a^x$ is a decreasing function

- Second derivative test

  $$\frac{d^2}{dx^2}a^x = \frac{d}{dx}(\ln a * a^x) = \ln^2 a * a^x$$ which is positive for all a.

  So a^x is a concave up function on its domain.

Examples 3: Sketch the graph of the following functions

a. $y = 2^x$ b. $y = 2^{-x}$

Sol.: a. $y = 2^x$

- **Symmetry:** $f(-x) = 2^{(-x)} = 2^{-x} \neq f(x)$

  $-f(x) = -2^x \neq f(-x)$

So the function has no symmetry.

- **Asymptote**: $\lim_{x\to\infty} 2^x = 2^\infty = \infty$; $\lim_{x\to-\infty} 2^x = 2^{-\infty} = \frac{1}{2^\infty} = \frac{1}{\infty} = 0,$

  so, $y = 0$ is horizontal asymptote

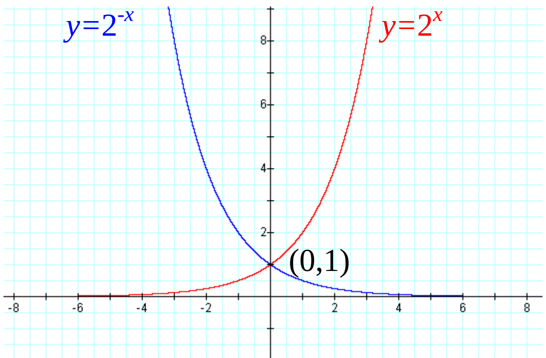
- $a = 2 > 1 \Rightarrow$ it is an increasing and concave up function.

b. $y = 2^{-x}$

- **Symmetry:** $f(-x) = 2^{-(-x)} = 2^x \neq f(x)$

  $-f(x) = -2^{-x} \neq f(-x)$

So the function has no symmetry.



- **Asymptote**: $\lim_{x\to\infty} 2^{-x} = 2^{-\infty} = 0$; $\lim_{x\to-\infty} 2^{-x} = 2^{-(-\infty)} = 2^{\infty} = \infty$,

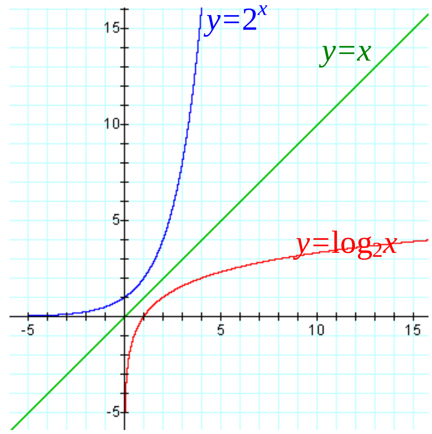
  so, $y = 0$ is horizontal asymptote

$y = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$ "x must be positive"

- $a = \frac{1}{2} < 1 \Rightarrow$ it is a decreasing and concave up function.

The Graph of $\log_a x$

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$.



Examples 4: Sketch the graph of the function $y = \log_2 x$

Sol.:

So $\lim_{x\to\infty} \log_2 x = \infty$, $\lim_{x\to 0^+} \log_2 x = -\infty$

Homework

I. Find dy/dx of the following functions:

1. $y = 7^x$
2. $y = 8^{x^2+1}$
3. $y = 9^{\sqrt{x}}$
4. $y = \log_{10}(x^4 + 3x^2 + 1)$
5. $y = \log_5 \sqrt{x^2 + 1}$
6. $y = \log_{10}\left|\frac{1-x^2}{2-5x^3}\right|$
7. $y = (x+1)^x$
8. $y = x^{\pi}.\pi^x$
9. $y = \ln \log_2 x$
10. $y = (3 + \sec^2 x)^{\frac{\tan^2 x}{(1+\sin x)^3}}$
11. $y = (1 + \cos^2 x)^{\frac{\sin^2 x}{(1+\cos x)^3}}$ (***Hint:*** use logarithmic differentiation)

II. Evaluate the following integrals:

1. $\int 10^{3x}.dx$
2. $\int 5^{-5x}.dx$
3. $\int x(3^{-x^2}).dx$
4. $\int \frac{(2^x + 1)}{2^x}.dx$
5. $\int \frac{3^x}{\sqrt{2^x + 4}}.dx$
6. $\int_{-1}^{1} 3^{3x-1}.dx$

7. $\int \frac{10^{\sqrt{x}}}{\sqrt{x}}.dx$ 8. $\int (3^x+3^{-x})^2.dx$ 9. $\int 4^x(4^x+1).dx$

10. $\int_{1/10}^{10} \frac{\log_{10}(10x)}{x}.dx$ 11. $\int_{0}^{9} \frac{2\log_{10}(x+1)}{x+1}.dx$ 12. $\int_{2}^{3} \frac{2\log_2(x-1)}{x-1}.dx$

13. $\int \frac{x+1}{x-\sqrt{x}}.dx$ 14. $\int \frac{e^{3x}+e^{2x}}{e^x-1}.dx$ 15. $\int \frac{\ln^3 x+1}{x\ln x-2x}.dx$

III. Find the limits of the following:

1. $\lim_{x\to\infty} \log_2 x$ 2. $\lim_{x\to\infty} 3^x$ 3. $\lim_{x\to\infty} 3^{-x}$

4. $\lim_{x\to 0^+} \log_{10} x$ 5. $\lim_{x\to 0^+} \log_{10}\left(\frac{1}{x}\right)$ 6. $\lim_{x\to\infty} \log_2\left(\frac{1}{x}\right)$

IV. Solve for x the following:

1. $3^{\log_3 7}+2^{\log_2 5}=5^{\log_5 x}$ 2. $8^{\log_8 3}+e^{\ln 5}=x^2-7^{\log_7 3x}$