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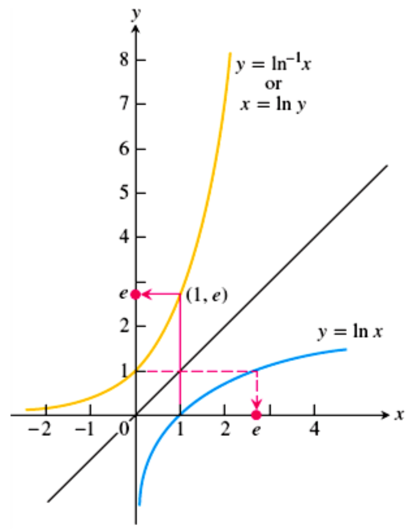
The Exponential Function:

Definition The Natural Exponential Function

For every real number x, $e^x = \ln^{-1} x = \exp(x)$ ($e = 2.718281828459045$)



Note1: Because e^x is the inverse of $\ln x$, its graph can be obtained by reflecting the graph of $\ln x$ across the line $y = x$.

So $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$

$\therefore$ The domain of e^x is the range of $\ln x = (-\infty, \infty)$

And the range of e^x is the domain of $\ln x = (0, \infty)$

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \text{ (for all } x > 0)$$

$$\ln(e^x) = x \text{ (for all } x \in R)$$

Note 2:

- To remove logarithms from an equation, take the exponential of both sides.
- To remove exponential from an equation, take the logarithm of both sides.

Laws of Exponents

For all real numbers a and b, the natural exponential obeys the following laws:

1. $e^a . e^b = e^{a+b}$ 2. $e^{-a} = \dfrac{1}{e^a}$

1. $\dfrac{e^a}{e^b} = e^{a-b}$ 4. $(e^a)^b = e^{a.b} = (e^b)^a$

To proof of the first law:

let $y_1 = e^a$ and $y_2 = e^b$

Then $\ln y_1 = \ln e^a = a$ and $\ln y_2 = \ln e^b = b$ (take the logarithm of both sides)

Add the two equations: $\ln y_1 + \ln y_2 = a + b$

$$\Rightarrow \quad \ln(y_1.y_2) = a + b \qquad \text{(exponentiate for both sides)}$$

$$\Rightarrow \quad e^{\ln(y_1.y_2)} = e^{a+b}$$

$$\therefore \quad y_1.y_2 = e^{a+b}$$

$$\text{So} \quad e^a.e^b = e^{a+b}$$

Examples: Solve for x the following:

1. $\ln(x+2)+\ln(x-3)=\ln 6$

Sol.: $\ln(x+2)(x-3)=\ln 6$

Exponentiate both sides:

$e^{\ln(x+2)(x-3)} = e^{\ln 6} \Rightarrow (x+2)(x-3)=6 \Rightarrow x^2-3x+2x-6=6 \Rightarrow x^2-x-12=0$

$\Rightarrow (x-4)(x+3)=0$

So either $x=4$ or $x = -3$ neglect (don't satisfy the equation)

2. $e^{2x} - 3e^x - 10 = 0$

Sol.: $(e^x-5)(e^x+2)=0$

So either $e^x=5 \Rightarrow x=\ln 5$

or $e^x = -2$ impossible

The Derivative of e^x:

Let $y = e^x$

$\therefore \ln y = x$ (take the logarithm of both sides)

$$\frac{d}{dx}(\ln y) = 1 \Rightarrow \frac{1}{y}.\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = y = e^x$$

$$\therefore \boxed{\frac{d}{dx}e^x = e^x}$$

In general:

$$\boxed{\frac{d}{dx}e^u = e^u.\frac{du}{dx}}$$ where u is a differential function of x.

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Examples 1: Find *dy/dx* of the following functions:

a. $y = e^{-x}$

Sol.: $\frac{dy}{dx} = e^{-x}(-1) = -e^{-x}$

b. $y = e^{x^2}$

Sol.: $\frac{dy}{dx} = e^{x^2}(2x) = 2xe^{x^2}$

c. $y = e^{\sin x}$

Sol.: $\frac{dy}{dx} = e^{\sin x} \cos x = \cos x e^{\sin x}$

Examples 2: Find d^2y/dx^2 of the following functions:

a. $y = e^{-x} \ln x$

Sol.: $\frac{dy}{dx} = e^{-x}.\frac{1}{x} - e^{-x} \ln x = e^{-x}(\frac{1}{x} - \ln x)$

$$\frac{d^2y}{dx^2} = e^{-x}(\frac{-1}{x^2} - \frac{1}{x}) - e^{-x}(\frac{1}{x} - \ln x) = e^{-x}(\frac{-1}{x^2} - \frac{2}{x} + \ln x)$$

$$= \frac{e^{-x}}{x^2}(-1 - 2x + x^2 \ln x)$$

b. $y = e^{-2x} \sin 3x$

Sol.: $\frac{dy}{dx} = e^{-2x}(3\cos 3x) - 2e^{-2x} \sin 3x = e^{-2x}(3\cos 3x - 2\sin 3x)$

$$\frac{d^2y}{dx^2} = e^{-2x}(-9\sin 3x - 6\cos 3x) - 2e^{-2x}(3\cos 3x - 2\sin 3x)$$

$$= e^{-2x}(-9\sin 3x - 6\cos 3x - 6\cos 3x + 4\sin 3x)$$

$$= e^{-2x}(-5\sin 3x - 12\cos 3x) = -e^{-2x}(5\sin 3x + 12\cos 3x)$$

The Integral of e^x:

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$$\int e^u .du = e^u + C$$

Examples: Evaluate the following integrals:

1. $\int e^{3x} dx$

Sol.: let $u = e^{3x} \Rightarrow du = 3e^{3x} dx$ ➔ $\therefore e^{3x} dx = \frac{du}{3}$

So $\int \frac{du}{3} = \frac{u}{3} + C = \frac{e^{3x}}{3} + C$

Another solution:

Let $u = 3x \Rightarrow du = 3dx$ ➔ $\therefore dx = \frac{du}{3}$

So $\int e^u \frac{du}{3} = \frac{e^u}{3} + C = \frac{e^{3x}}{3} + C$

2. $\int e^{\sin x} . \cos x dx$

Sol.: let $u = \sin x \Rightarrow du = \cos x dx$ ➔ $\therefore \cos x dx = du$

So $\int e^u du = e^u + C = e^{\sin x} + C$

3. $\int e^{3x+1} dx$

Sol.: let $u = 3x + 1 \Rightarrow du = 3dx$ ➔ $\therefore dx = \frac{du}{3}$

So $\int e^u \frac{du}{3} = \frac{e^u}{3} + C = \frac{e^{3x+1}}{3} + C$

4. $\int x e^{x^2 - 3} dx$

Sol.: let $u = x^2 - 3 \Rightarrow du = 2x dx$ ➔ $\therefore x dx = \frac{du}{2}$

So $\int e^u \frac{du}{2} = \frac{e^u}{2} + C = \frac{e^{x^2 - 3}}{2} + C$

5. $\int \frac{e^x}{e^x - 1} dx$

Sol.: let $u = e^x - 1 \quad \Rightarrow \quad du = e^x dx$ ➔ $\therefore xdx = \frac{du}{2}$

So $\int \frac{du}{u} = \ln|u| + C = \ln|e^x - 1| + C$

6. $\int e^{x+e^x} dx = \int e^x * e^{e^x} dx$

Sol.: let $u = e^x \quad \Rightarrow \quad du = e^x dx$

So $\int e^u du = e^u + C = e^{e^x} + C$

Homework:

I. Find dy/dx of the following:

1. $y = e^{-5x^2}$
2. $y = e^{\sqrt{1+5x^3}}$
3. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $y = \frac{e^x}{\ln x}$
5. $y = \ln \cos e^x$
6. $y = \ln(1 - xe^{-x})$
7. $y = e^{x \tan x}$
8. $y = e^{x - e^{3x}}$
9. $y = \sin(e^x)$

II. Evaluate the following integrals:

1. $\int e^{2x} dx$
2. $\int \frac{e^x}{1+e^x} dx$
3. $\int x^3 e^{x^4} dx$
4. $\int \frac{e^{\sqrt{y+1}}}{\sqrt{y+1}} dy$
5. $\int \frac{dy}{\sqrt{y}.e^{\sqrt{y}}}$
6. $\int_0^e \frac{dx}{x+e}$
7. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$
8. $\int \frac{dx}{e^x}$
9. $\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x + 4} dx$
10. $\int_1^{\sqrt{2}} xe^{-x^2} dx$
11. $\int e^{2\ln x} dx$
12. $\int [\ln(e^x) - \ln(e^{-x})] dx$
13. $\int_{e^{-6}}^{e^6} \frac{\sqrt{36 - \ln x}}{x} dx$
14. $\int \frac{\sec x dx}{\sqrt{\ln(\sec x + \tan x)}}$

Example 1: Generate or sketch the graph of $y = e^{-x^2/2}$.

Sol.:

1. **Domain:** $D_f = (-\infty, \infty)$.
2. **Symmetry:** $f(-x) = e^{-(-x)^2/2} = e^{-x^2/2} = f(x) \quad o.k.$

$$-f(x) = -e^{-x^2/2} \neq f(-x) \quad not \quad o.k.$$

So it is an even function (It has symmetry about y-axis).

3. x and y-intercept:

When $x = 0 \Rightarrow y = e^{-0^2/2} = e^0 = 1$ $\therefore (0,1)$ is y-intercept.

Assume $y = 0 \Rightarrow e^{-x^2/2} \neq 0$ So there is no x-intercept.

4. Asymptotes:

- Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{-x^2/2} = \lim_{x\to\infty} e^{-\infty^2/2} = e^{-\infty} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{-x^2/2} = e^{-(-\infty)^2/2} = e^{-\infty} = 0$$

$\therefore y = 0$ (x-axis) is horizontal asymptote

- Vertical asymptotes:

There is no vertical asymptote because that $f(x) \neq \infty$ at $x=a$.

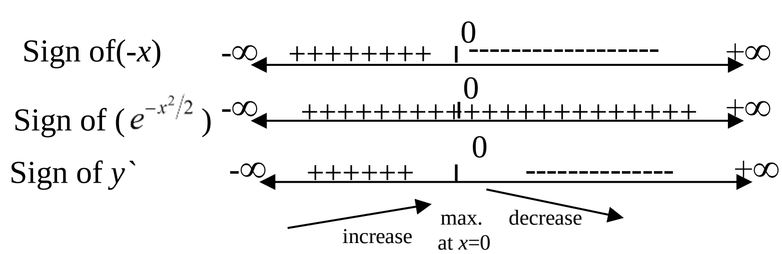
5. First derivative test:

$$\frac{dy}{dx} = e^{-x^2/2} * \frac{-2x}{2} = -xe^{-x^2/2}$$

Put $\frac{dy}{dx} = 0 \Rightarrow -xe^{-x^2/2} = 0$ since $e^{-x^2/2} \neq 0$

$\therefore$ at $x = 0$ there is a critical point.

Note: Sign of $-xe^{-x^2/2}$ depend only on (-x) because that $e^{-x^2/2} > 0$ for all x.



So $(0,1)$ is max. point.

The function increases on $(-\infty,0]$.

The function decreases on $[0,\infty)$.

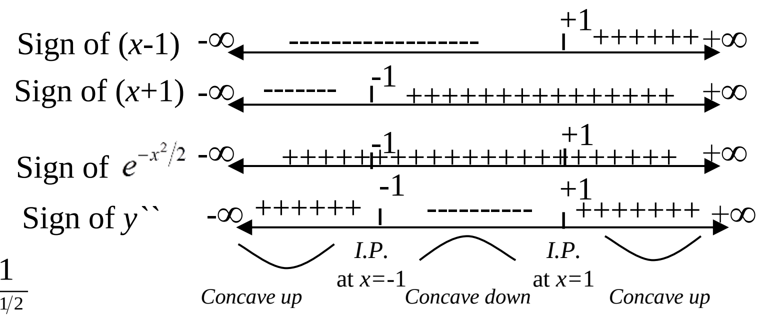
6. **Second derivative test:**

$$\frac{d^2y}{dx^2} = -x\left[-xe^{-x^2/2}\right] - e^{-x^2/2} = (x^2-1)e^{-x^2/2}$$

Put $\frac{d^2y}{dx^2} = 0 \Rightarrow (x^2-1)e^{-x^2/2} = 0$

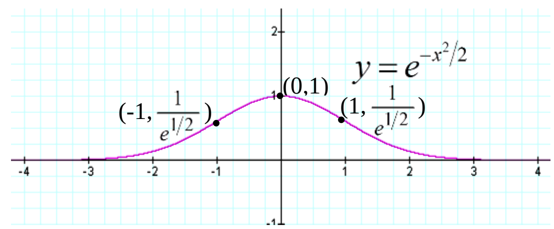
since $e^{-x^2/2} \neq 0 \Rightarrow (x^2-1) = 0$

$\Rightarrow (x-1)(x+1) = 0$



When $x = -1 \Rightarrow y = e^{-(-1)^2/2} = e^{-1/2} = \frac{1}{e^{1/2}}$

and $x = 1 \Rightarrow y = e^{-(1)^2/2} = e^{-1/2} = \frac{1}{e^{1/2}}$



So the curve has inflection points at $(-1, \frac{1}{e^{1/2}})$ and $(1, \frac{1}{e^{1/2}})$

It concaves up on $(-\infty,-1)$ and $(1, \infty)$,

And concaves down on $(-1, 1)$

So $R_f = (0,1]$

Example 2: Generate or sketch the graph of $y = \frac{\ln x}{x}$.

Sol.:

1. **Domain:** $D_f = (0, \infty)$.
2. **Symmetry:** Because $x > 0$ for all x, the function has no symmetry.
3. ***x* and *y*-intercept:**

   When $x \neq 0 \Rightarrow$ So there is no *y*-intercept

   Assume $y = 0 \Rightarrow \frac{\ln x}{x} = 0 \Rightarrow \ln x = 0 \Rightarrow x = e^0 = 1 \therefore (1,0)$ is *x*-intercept.

4. **Asymptotes:**

   a. Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$\lim_{x\to-\infty} f(x)$ impossible because $x > 0$.

$\therefore y = 0$ (x-axis) is horizontal asymptote.

b. Vertical asymptotes:

$$\lim_{x\to o^+} f(x) = \lim_{x\to o^+} \frac{\ln x}{x} = \frac{-\infty}{0} = -\infty$$

$\therefore x = 0$ (y-axis) is vertical asymptote.

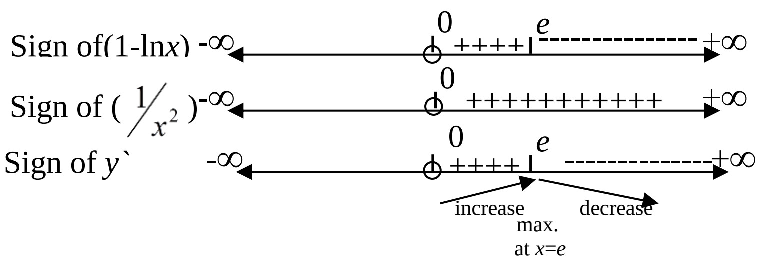
5. First derivative test:

$$\frac{dy}{dx} = \frac{-\ln x}{x^2} + \frac{1}{x} * \frac{1}{x} = \frac{1}{x^2}(1-\ln x)$$

Put $\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{1}{x^2}(1-\ln x) = 0$

$\therefore (1-\ln x) = 0 \quad \Rightarrow \quad \ln x = 1 \quad \Rightarrow$

$x = e^1 = e$



$\therefore$ at $x = e$ there is a critical point. $\therefore y = \frac{\ln e}{e} = \frac{1}{e} = e^{-1}$

So (e, e^{-1}) is max. point.

The function increases on $(0, e]$

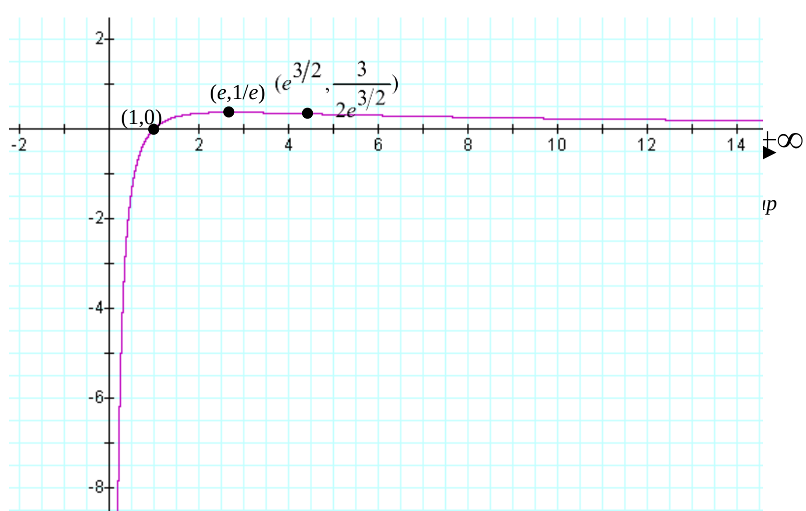
The function decreases on $[e, \infty)$

6. Second derivative test:

$$\frac{d^2y}{dx^2} = \frac{-2}{x^3}(1-\ln x) - \frac{1}{x^3} = \frac{-1}{x^3}(2-2\ln$$

$$= \frac{2\ln x - 3}{x^3}$$

Put $\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \frac{2\ln x - 3}{x^3} = 0$



since $x^3 \neq 0 \quad \Rightarrow \quad 2\ln x - 3 = 0$

$\Rightarrow \quad \ln x = \frac{3}{2} \quad \Rightarrow \quad x = e^{3/2}$

$\Rightarrow \quad y = \frac{\ln e^{3/2}}{e^{3/2}} = \frac{3/2}{e^{3/2}}$

Sign of $y``$ depends on $(2\ln x\text{-}3)$

So the curve has inflection points at $(e^{3/2}\ ,\ \frac{3}{2e^{3/2}}\ )$

It concaves down on $(0,\ e^{3/2}\ )$ and concaves up on $(e^{3/2}\ ,\ \infty)$. $\therefore R_f = (-\infty, \frac{1}{e}]$.

Homework: Sketch the graph of the following functions:

1. $y = xe^{x^2}$
2. $y = \frac{e^x}{x}$
3. $y = x^2 e^{-2x}$
4. $y = x\ln x$
5. $y = \frac{\ln x}{x^2}$
6. $y = x^2 \ln(2x)$