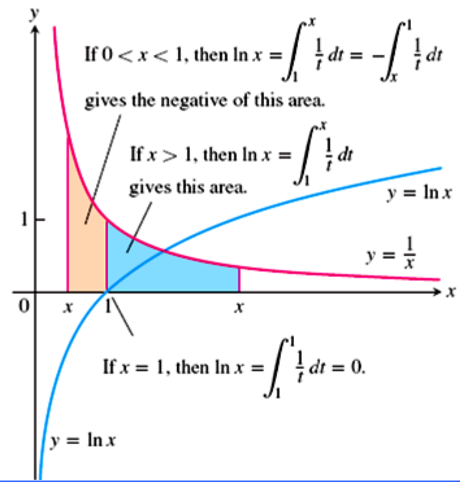
syllabus

Logarithmic Functions:

The **Natural Logarithm** of positive number *x*, written as ln *x*, is the value of integral:

$$\ln x = \int_1^x \frac{1}{t}\,dt \; ; \; x > 0$$



We can see from the figure

- $\ln x > 0$ if $x > 1$
- $\ln x = 0$ if $x = 1$
- $\ln x < 0$ if $0 < x < 1$.

Definition The number e

The number e is that the number in the domain of the natural logarithm satisfy

$$\ln(e) = 1$$

Geometrically, the number e corresponds to the point on the x-axis for which the area under graph of ($y=1/t$) and above the interval $[1, e]$ is the exact area of the unit square.

Rules of arithmetic for logarithms

For any positive numbers a and b and for any exponent n,

1. $\ln ab = \ln a + \ln b$
2. $\ln \frac{a}{b} = \ln a - \ln b$
3. $\ln \frac{1}{b} = -\ln b$ (Rule 2 with a=1)
4. $\ln a^n = n \ln a$
5. $\ln \sqrt[n]{a} = \frac{1}{n} \ln a$

The derivative of $y = \ln x$

By the First Fundamental Theorem of Calculus

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

If u is a differentiable function of x whose values are positive,

$$\frac{d}{dx}\ln u = \frac{1}{u}.\frac{du}{dx}$$

Examples: Find $\frac{dy}{dx}$ for the following functions:

1. $y = \ln(3x^2+4)$

Sol.: $\frac{dy}{dx} = \frac{1}{3x^2+4}*(2*3x) = \frac{6x}{3x^2+4}$

2. $y = \ln^3 x = (\ln x)^3$

Sol.: $\frac{dy}{dx} = 3(\ln x)^2 * \frac{1}{x}$

3. $y = \ln x^3 = 3\ln x$

Sol.: $\frac{dy}{dx} = \frac{3}{x}$

4. $y = \ln\sqrt{\cos^3 x} = \ln\cos^{\frac{3}{2}} x = \frac{3}{2}\ln\cos x$

Sol.: $\frac{dy}{dx} = \frac{3}{2}*\frac{1}{\cos x}(-\sin x) = \frac{-3}{2}\tan x$

5. $y = x[\sin(\ln x)+\cos(\ln x)]$

Sol.: $\frac{dy}{dx} = x[\cos(\ln x).\frac{1}{x} - \sin(\ln x).\frac{1}{x}] + [\sin(\ln x)+\cos(\ln x)]$

$$= \cos(\ln x) - \sin(\ln x) + \sin(\ln x) + \cos(\ln x)$$

$$= 2\cos(\ln x)$$

6. $y = \ln\frac{x\sqrt{x-2}}{(x-2)^2}$

Sol.: Simplify the right-hand side

$$y = \ln x + \ln\sqrt{x-2} - \ln(x-2)^2 \quad \Rightarrow \quad y = \ln x + \frac{1}{2}\ln(x-2) - 2\ln(x-2)$$

$$y = \ln x - \frac{3}{2}\ln(x-2)$$

Differentiate:

$$\frac{dy}{dx} = \frac{1}{x} - \frac{3}{2}*\frac{1}{x-2}$$

$$\frac{dy}{dx}=\frac{1}{x}-\frac{3}{2(x-2)}$$

Logarithmic Differentiation:

The derivatives of positive function given by formulas that involve products, quotients and powers can often be found more quickly if we take natural logarithm of both sides before differentiating. This enables us to use laws of logarithms to simplify the formulas before differentiating. The process is called ***logarithmic differentiation.***

Examples: Find $\frac{dy}{dx}$ for the following functions:

1. $y=\frac{(x^2+1)(x+3)^{1/2}}{x-1}$

Sol.: We take the natural logarithm of both sides and simplify the result with properties of logarithms:

$$y=\frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$\ln y=\ln\left[\frac{(x^2+1)(x+3)^{1/2}}{x-1}\right]=\ln[(x^2+1)(x+3)^{1/2}]-\ln(x-1)$$

$$=\ln(x^2+1)+\ln(x+3)^{1/2}-\ln(x-1)$$

$$=\ln(x^2+1)+\frac{1}{2}\ln(x+3)-\ln(x-1)$$

Then we differentiate by implicit differentiation

$$\frac{1}{y}*\frac{dy}{dx}=\frac{2x}{x^2+1}+\frac{1}{2}*\frac{1}{x+3}-\frac{1}{x-1}$$

$$\frac{dy}{dx}=y\left[\frac{2x}{x^2+1}+\frac{1}{2x+6}-\frac{1}{x-1}\right]$$

$$\frac{dy}{dx}=\left[\frac{(x^2+1)(x+3)^{1/2}}{x-1}\right]\left[\frac{2x}{x^2+1}+\frac{1}{2x+6}-\frac{1}{x-1}\right]$$

2. $y^{2/3}=\dfrac{(x^2+1)(3x+4)^{1/2}}{\sqrt[5]{(2x-3)(x^2-4)}}$

Sol.: We take the natural logarithm of both sides and simplify the result with properties of logarithms:

$$\ln y^{2/3}=\ln\left[\frac{(x^2+1)(3x+4)^{1/2}}{\sqrt[5]{(2x-3)(x^2-4)}}\right]$$

$$\frac{2}{3}\ln y=\ln[(x^2+1)(3x+4)^{1/2}]-\ln\sqrt[5]{(2x-3)(x^2-4)}$$

$$\frac{2}{3}\ln y=\ln(x^2+1)+\ln(3x+4)^{1/2}-\frac{1}{5}\ln[(2x-3)(x^2-4)]$$

$$\frac{2}{3}\ln y=\ln(x^2+1)+\frac{1}{2}\ln(3x+4)-\frac{1}{5}\ln(2x-3)-\frac{1}{5}\ln(x^2-4)$$

Then we differentiate by implicit differentiation

$$\frac{2}{3}*\frac{1}{y}*\frac{dy}{dx}=\frac{2x}{x^2+1}+\frac{1}{2}*\frac{3}{3x+4}-\frac{1}{5}*\frac{2}{2x-3}-\frac{1}{5}*\frac{2x}{x^2-4}$$

$$\therefore\ \frac{dy}{dx}=\frac{3}{2}y\left[\frac{2x}{x^2+1}+\frac{1}{2}*\frac{3}{3x+4}-\frac{1}{5}*\frac{2}{2x-3}-\frac{1}{5}*\frac{2x}{x^2-4}\right]$$

$$\frac{dy}{dx}=\frac{3}{2}\left[\frac{(x^2+1)(3x+4)^{1/2}}{\sqrt[5]{(2x-3)(x^2-4)}}\right]^{3/2}\left[\frac{2x}{x^2+1}+\frac{3}{6x+8}-\frac{2}{10x-15}-\frac{2x}{5x^2-20}\right]$$

3. $y=\sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Sol.: We take the natural logarithm of both sides and simplify the result with properties of logarithms:

$$\ln y=\ln\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$$

$$\ln y=\frac{1}{3}\ln\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}$$

$$\ln y=\frac{1}{3}\left[\ln x+\ln(x+1)+\ln(x-2)-\ln(x^2+1)-\ln(2x+3)\right]$$

Then we differentiate by implicit differentiation

$$\frac{1}{y}*\frac{dy}{dx}=\frac{1}{3}\left[\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x-2}-\frac{2x}{x^2+1}-\frac{2}{2x+3}\right]$$

$$\frac{dy}{dx}=\frac{y}{3}\left[\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x-2}-\frac{2x}{x^2+1}-\frac{2}{2x+3}\right]$$

$$\frac{dy}{dx}=\frac{1}{3}\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}\left[\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x-2}-\frac{2x}{x^2+1}-\frac{2}{2x+3}\right]$$

The Integral $\int \frac{1}{u}\,du$

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\,du=\ln|u|+C$$

Examples: Evaluate the following integrals:

1. $\int \frac{x}{3x^2+4}\,dx$

Sol.: Let $u=3x^2+4 \Rightarrow du=6x.dx$

$\therefore x.dx=\frac{du}{6}$

So $\int \frac{xdx}{3x^2+4}=\int \frac{1}{u}*\frac{du}{6}=\frac{1}{6}\int \frac{du}{u}=\frac{1}{6}\ln|u|+C=\frac{1}{6}\ln|3x^2+4|+C$

2. $\int_0^2 \frac{2x}{x^2-5}\,dx$

Sol.: Let $u=x^2-5 \Rightarrow du=2x.dx$

$\therefore 2x.dx=du$

When $x=0 \Rightarrow u(0)=(0)^2-5=-5$

And $x=2 \Rightarrow u(2)=(2)^2-5=-1$

So $\displaystyle\int_0^2 \frac{2x}{x^2-5}dx = \int_{-5}^{-1}\frac{du}{u} = \ln|u|\Big|_{-5}^{-1} = \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$

3. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}d\theta$

Sol.: Let $u = 3+2\sin\theta \quad\Rightarrow\quad du = 2\cos\theta d\theta$

$\therefore 4\cos\theta d\theta = 2.du$

When $\theta = -\dfrac{\pi}{2} \Rightarrow u(-\dfrac{\pi}{2}) = 3+2\sin(-\dfrac{\pi}{2}) = 3-2 = 1$

And $\theta = \dfrac{\pi}{2} \Rightarrow u(\dfrac{\pi}{2}) = 3+2\sin(\dfrac{\pi}{2}) = 3+2 = 5$

So $\displaystyle\int_{-\pi/2}^{\pi/2}\frac{4\cos\theta}{3+2\sin\theta}d\theta = \int_1^5 \frac{2.du}{u} = 2\ln|u|\Big|_1^5 = 2[\ln|5| - \ln|1|] = 2\ln 5$

4. $\displaystyle\int \frac{5x+2}{x+1}dx$

Sol.: For rational functions when the degree of denominator is equal or greater the degree of numerator, us the long division to simplify the integral.

$$\begin{array}{r}5\\ x+1\overline{)\;5x+2}\\ \underline{\mp 5x \mp 5}\\ -3\end{array}$$

$\displaystyle\int \frac{5x+2}{x+1}dx = \int\left(5 - \frac{3}{x+1}\right).dx = 5x - 3\ln|x+1| + C$

5. $\displaystyle\int \frac{x^2+2x+2}{x+2}dx$

$$\begin{array}{r}x\\ x+2\overline{)\;x^2+2x+2}\\ \underline{\mp x^2 \mp 2x}\\ 2\end{array}$$

Sol.: $\displaystyle\int \frac{x^2+2x+2}{x+2}dx = \int\left(x + \frac{2}{x+2}\right).dx = \frac{x^2}{2} + 2\ln|x+2| + C$

6. $\displaystyle\int \frac{dx}{x\ln x}$

Sol.: Let $u = \ln x \quad\Rightarrow\quad du = \dfrac{dx}{x}$

$\displaystyle\therefore \int \frac{dx}{x\ln x} = \int\frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$

7. $\int \frac{dx}{\sqrt{x}(1-\sqrt{x})}$

Sol.: Let $u=\sqrt{x} \quad \Rightarrow u^2=x \quad \Rightarrow \quad 2udu=dx$

$$\therefore \int \frac{dx}{\sqrt{x}(1-\sqrt{x})}=\int \frac{2udu}{u(1-u)}=\int \frac{2du}{(1-u)}=2\int \frac{du}{(1-u)}$$

Let $z=1-u \quad \Rightarrow dz=-du \quad \Rightarrow \quad du=-dz$

$$\therefore 2\int \frac{du}{(1-u)}=2\int \frac{-dz}{z}=-2\ln|z|+C=-2\ln|1-u|+C=-2\ln|1-\sqrt{x}|+C$$

The Integral of tan *x*, cot *x*, sec*x* and csc*x*:

Examples: Evaluate the following integrals,

1. $\int \tan x.dx=\int \frac{\sin x}{\cos x}dx$

Let $u=\cos x \quad \Rightarrow \quad du=-\sin xdx$ or $\sin xdx=-du$

$$\int \frac{\sin x}{\cos x}dx=\int \frac{-du}{u}=-\ln|u|+C=-\ln|\cos x|+C$$

$$=\ln|(\cos x)^{-1}|+C=\ln\left|\frac{1}{\cos x}\right|+C=\ln|\sec x|+C$$

2. $\int \cot x.dx=\int \frac{\cos x}{\sin x}dx$

Let $u=\sin x \quad \Rightarrow \quad du=\cos x$

$$\int \frac{\cos x}{\sin x}dx=\int \frac{du}{u}=\ln|u|+C=\ln|\sin x|+C$$

3. $\int \sec x.dx=\int \sec x.dx*\frac{\sec x+\tan x}{\sec x+\tan x}$ (multiply and divide by $\sec x+\tan x$)

$$\int \sec x.dx*\frac{\sec x+\tan x}{\sec x+\tan x}=\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}.dx$$

Let $u=\sec x+\tan x \Rightarrow du=(\sec x\tan x+\sec^2 x).dx$

$$\int \frac{(\sec x\tan x+\sec^2 x).dx}{\sec x+\tan x}=\int \frac{du}{u}=\ln|u|+C=\ln|\sec x+\tan x|+C$$

$$\therefore \int \sec x.dx = \ln|\sec x + \tan x| + C$$

And in the same manner we can show that $\int \csc x.dx = -\ln|\csc x + \cot x| + C$

So

$$\begin{aligned}
&\int \tan u du = -\ln|\cos x| = \ln|\sec u| + C \\
&\int \cot u du = \ln|\sin u| + C = -\ln|\csc u| + C \\
&\int \sec u.du = \ln|\sec u + \tan u| + C \\
&\int \csc u.du = -\ln|\csc u + \cot u| + C
\end{aligned}$$

Example: Evaluate $\int_0^{\pi/6} \tan 2x dx$

Sol.: Let $u = 2x \;\Rightarrow\; du = 2dx$ or $dx = \dfrac{du}{2}$

When $x = 0 \;\Rightarrow\; u(0) = 2*0 = 0$

$$x = \frac{\pi}{6} \;\Rightarrow\; u(\frac{\pi}{6}) = 2*\frac{\pi}{6} = \frac{\pi}{3}$$

$$\int_0^{\pi/6} \tan 2x dx = \int_0^{\pi/3} \tan u.\frac{du}{2} = \frac{1}{2}\ln|\sec u|\Big|_0^{\pi/3} = \frac{1}{2}\left[\ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0|\right]$$

$$= \frac{1}{2}[\ln 2 - \ln 1] = \frac{1}{2}\ln 2$$