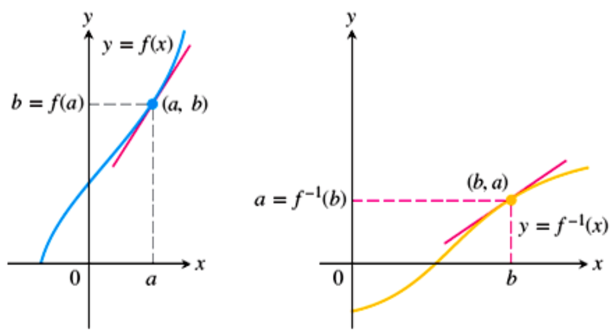
Derivatives of Inverse Differentiable Functions:

Theorem: The Derivative Rule for Inverses

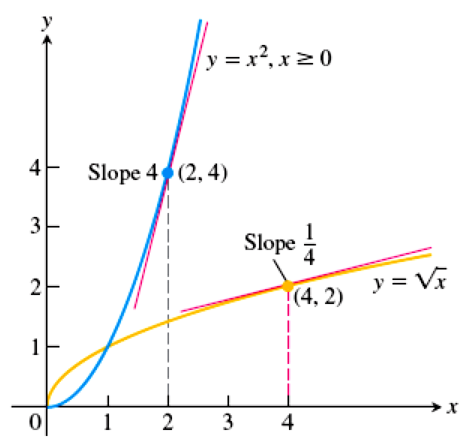
If f has an interval I as domain and $f\,'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $\frac{df^{-1}}{dx}$ at a point b in the domain of f^{-1} is the reciprocal of the value of f ` at the point $a=f^{-1}(b)$:

$$\left.\frac{df^{-1}}{dx}\right|_{x=b=f(a)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=a}} \quad \ldots(1)$$



The slopes are reciprocal: $(f^{-1})'(b) = \frac{1}{f'(a)}$ or $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

Example 6: Verify Eq.(1) for $f(x) = x^2$, $x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ at the point $x = 2$ in the domain of f.



Sol.: at $x = 2 \Rightarrow f(2) = 2^2 = 4$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(a)} = \left.\frac{d}{dx}\sqrt{x}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{2*2} = \frac{1}{4}$$

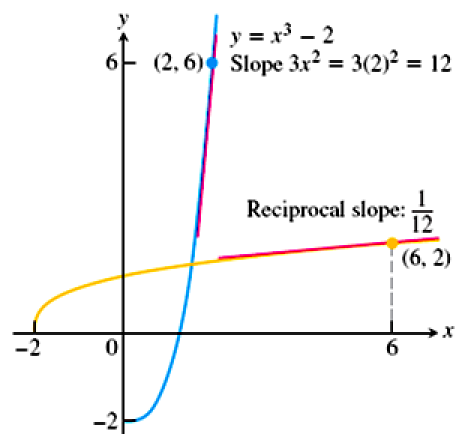
$$\left.\frac{df}{dx}\right|_{x=2} = \left.\frac{d}{dx}x^2\right|_{x=2} = 2x|_{x=2} = 2*2 = 4$$

Thus $\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}}$

Example 7: Let $f(x) = x^3 - 2$. Find the value of $\frac{df^{-1}}{dx}$ at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Sol.: $\left.\frac{df}{dx}\right|_{x=2} = \left.\frac{d}{dx}(x^3-2)\right|_{x=2} = \left.3x^2\right|_{x=2} = 3(2)^2 = 3*4 = 12$

Thus $\left.\frac{df^{-1}}{dx}\right|_{x=f(2)=6} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12}$.



Homework

1. Find a formula for the inverse of the function and show that $f(f^{-1}(x))=f^{-1}(f(x))$.

a. $y = \sqrt{10-3x}$ b. $y = \frac{4x-1}{2x+3}$ c. $y = 2x^3+3$

d. $y = 1-\frac{2}{x^2}; x>0$ e. $y = \sqrt{x^2+2x}; x>0$ f. $y = \frac{1}{x^3}; x \neq 0$

g. $y = (x+1)^2; x \geq 1$ h. $y = x^{2/3}; x \geq 0$ i. $y = (1/2)x - 7/2$

2. Find a formula for the inverse of the function and verify that $\left.\frac{df^{-1}}{dx}\right|_{x=f(a)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=a}}$

by evaluating df/dx at $x=a$ and df^{-1}/dx at $x=f(a)$.

a. $f(x) = 2x + 3$, $a = -1$ b. $f(x) = 5-4x$, $a = 1/2$

c. $f(x) = (1/5)x + 7$, $a = -1$ d. $f(x) = 2x^2$, $x>0$, $a = 5$

[syllabus]

Mathematics