Area of Surface of Revolution:

If the function $y=f(x) > 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y=f(x)$ about the x-axis is calculated as following:

The surface area of typical cylinder is $dS=2\pi r.dL$

dL will be calculated from one of the following three relations:

i. $dL = \sqrt{1+\left(\frac{dy}{dx}\right)^2}.dx$

ii. $dL = \sqrt{1+\left(\frac{dx}{dy}\right)^2}.dy$

iii. $dL = \sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}.dt$

r or ρ is the radius of the typical cylinder: (As in this case when the curve is rotated about x-axis), then

$$r=y=f(x)$$

So the surface area: $S = \int dS = \int_a^b 2\pi.r.dL$

If we represent dL by the first equation, then:

$$S = \int_a^b 2\pi.y.\sqrt{1+\left(\frac{dy}{dx}\right)^2}.dx = \int_a^b 2\pi.f(x).\sqrt{1+[f`(x)]^2}.dx \qquad \text{----(1)}$$

When the same area is rotated about y-axis then:

$$r=x$$

The surface area is

$$S = \int_a^b 2\pi.x.\sqrt{1+[f`(x)]^2}.dx \qquad \text{----(2)}$$

Note: We can use this expression instead of equation (1) in case of the curve is expressed as $x=f(y)$

$$S=\int_c^d 2\pi.y.\sqrt{1+\left(\frac{dx}{dy}\right)^2}.dy=\int_c^d 2\pi.y.\sqrt{1+[f`(y)]^2}.dy \qquad \text{----(3)}$$

and this expression instead of equation (2) in case of the curve is expressed as $x=f(y)$

$$S=\int_c^d 2\pi.x.\sqrt{1+\left(\frac{dx}{dy}\right)^2}.dy=\int_c^d 2\pi.f(y).\sqrt{1+[f`(y)]^2}.dy \qquad \text{----(4)}$$

If the curve is expressed as parametric equation such:

$x=x(t),\ y=y(t) \qquad a\le t\le b$

and $\frac{dx}{dt}$, $\frac{dx}{dt}$ are both continuous in above interval then the area of surface area generated by revolving this curve

i. about x-axis is

$$S=\int_a^b 2\pi.y(t)\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}.dt \qquad \text{----(5)}$$

ii. about y-axis is

$$S=\int_a^b 2\pi.x(t)\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}.dt \qquad \text{----(6)}$$

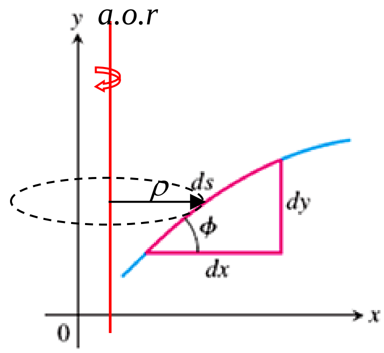
Or in general from short differential form

$$S=\int dS=\int 2\pi.\rho.dL$$

Where $dL=\sqrt{dx^2+dy^2}$

and ρ: is the radius from axis of revolution to an element of arc-length dL. If axis of rotation is

- $x=k$ then $\rho=x\text{-}k$
- $y=k$ then $\rho=y\text{-}k$



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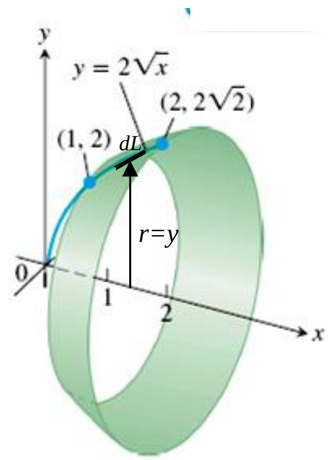
Example 1: Find the area of the surface generated by revolving the curve $y=2\sqrt{x}$, $1\le x\le 2$ about x-axis.

Sol.: $dS=2\pi r.dL$



where $r=y=2\sqrt{x}$

and $dL=\sqrt{1+[f`(x)]^2}.dx$

$$f`(x)=2*\frac{1}{2}x^{-1/2}=\frac{1}{\sqrt{x}}$$

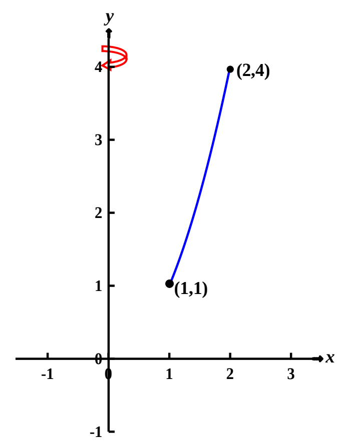
$$dS=2\pi(2\sqrt{x})\sqrt{1+[f`(x)]^2}.dx$$

$$=4\pi\sqrt{x}\sqrt{1+[\frac{1}{\sqrt{x}}]^2}.dx=4\pi\sqrt{x}\sqrt{1+\frac{1}{x}}.dx$$

$$=4\pi\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}.dx=4\pi\sqrt{x+1}.dx \quad ➔ \quad \therefore S=\int dS=\int_1^2 4\pi\sqrt{x+1}.dx=4\pi\frac{(x+1)^{3/2}}{3/2}\bigg|_1^2$$

$$=\frac{8\pi}{3}[(2+1)^{3/2}-(1+1)^{3/2}]\approx 19.836 \text{ square units}$$

Example 2: Find the area of the surface generated by revolving the portion of the curve $y=x^2$ between $x=1$ and $x=2$ about y-axis.



Sol.: $dS=2\pi r.dL$

where $r=x$

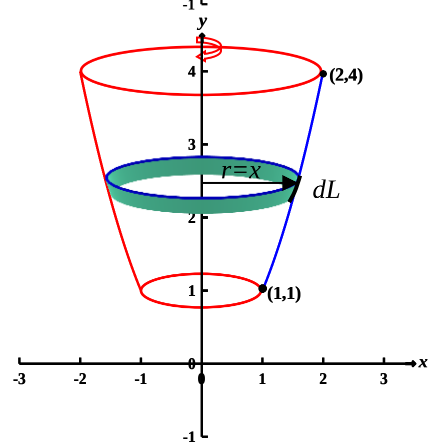
and $dL=\sqrt{1+[f`(x)]^2}.dx$

$y=x^2 \Rightarrow \quad f`(x)=2x$

$\therefore dL=\sqrt{1+(2x)^2}.dx=\sqrt{1+4x^2}.dx$

$$\therefore S=\int dS=\int_1^2 2\pi.x\sqrt{1+4x^2}.dx=2\pi[\frac{1}{8}\frac{(1+4x^2)^{3/2}}{3/2}\bigg|_1^2$$

$$=\frac{\pi}{6}(1+4x^2)^{3/2}\bigg|_1^2=\frac{\pi}{6}[(1+4*2^2)^{3/2}-(1+4*1^2)^{3/2}]$$



$$=\frac{\pi}{6}[17^{3/2}-5^{3/2}]\approx 30.85 \text{ square units.}$$

Another solution: Use $x=f(y)$

$$y=x^2 \Rightarrow \quad x=\sqrt{y} \quad \Rightarrow \quad \frac{dx}{dy}=\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2=\left(\frac{1}{2\sqrt{y}}\right)^2=\frac{1}{4y}$$

$$\therefore dL=\sqrt{1+\left(\frac{dx}{dy}\right)^2}.dy=\sqrt{1+\frac{1}{4y}}.dy=\sqrt{\frac{4y+1}{4y}}.dy=\frac{\sqrt{4y+1}}{2\sqrt{y}}.dy$$

The limits of integration:

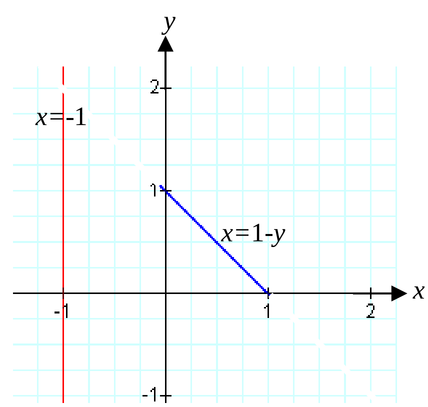
When $x=1 \Rightarrow y=(1)^2=1$ and when $x=2 \Rightarrow y=(2)^2=4$

$$\therefore S=\int dS=\int_1^4 2\pi.x\frac{\sqrt{4y+1}}{2\sqrt{y}}.dy=\int_1^4 2\pi.\sqrt{y}\frac{\sqrt{4y+1}}{2\sqrt{y}}.dy$$

$$=\int_1^4 \pi.\sqrt{4y+1}dy=\pi*\frac{1}{4}\frac{(4y+1)^{3/2}}{3/2}\Bigg|_1^4=\frac{\pi}{3}[(4*4+1)^{3/2}-(4*1+1)^{3/2}]$$

$$=\frac{\pi}{6}[17^{3/2}-5^{3/2}]\approx 30.58 \text{ square units.}$$

Example 3: The line segment $x=1-y$, $0\le y\le 1$, is revolve about $x=-1$ to generate truncated cone. Find its lateral surface area (which excludes the top and base areas).



Sol.: $dS=2\pi r dL$

$r=x-k=x-(-1)=x+1$

$$dL=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.dx$$

and $\frac{dy}{dx}=\frac{d}{dx}(1-x)=-1$

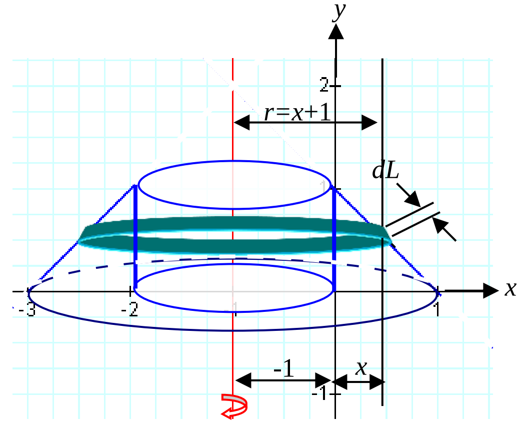
$$\Rightarrow\ 1+\left(\frac{dy}{dx}\right)^2 = 1+(-1)^2 = 1+1=2$$

$$\therefore dL=\sqrt{2}.dx$$

So $dS = 2\pi r dL = 2\pi(x+1)\sqrt{2}dx$

When $y=0 \quad \Rightarrow\ x=1\text{-}0=1$

$\qquad\ \ y=1 \quad \Rightarrow\ x=1\text{-}1=0$



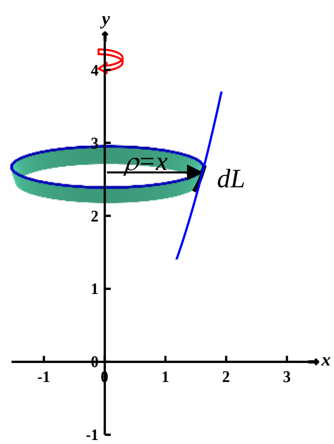
$$\therefore S=\int dS = 2\sqrt{2}\pi\int_0^1 (x+1).dx = 2\sqrt{2}\pi\left[\frac{x^2}{2}+x\right]_0^1$$

$$=2\sqrt{2}\pi\left[\left(\frac{1^2}{2}+1\right)-\left(\frac{0^2}{2}+0\right)\right]=2\sqrt{2}\pi\left(\frac{3}{2}\right)=3\sqrt{2}\pi \text{ square units}$$

Example 4: Find the area of the surface generated by revolving the parametric curve $x=\cos^2 t$, $y=\sin^2 t$, $0 \le t \le \pi/2$ about y-axis.

Sol.: $\quad dS=2\pi\rho.dL$

where $\rho = x = \cos^2 t$



and $dL=\sqrt{dx^2+dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}.dt$

$$x=\cos^2 t \ \Rightarrow\ \frac{dx}{dt} = -2\cos t\sin t \ \Rightarrow\ \left(\frac{dx}{dt}\right)^2 = 4\cos^2 t\sin^2 t$$

$$y=\sin^2 t \ \Rightarrow\ \frac{dy}{dt} = 2\sin t\cos t \ \Rightarrow\ \left(\frac{dy}{dt}\right)^2 = 4\sin^2 t\cos^2 t$$

$$dL=\sqrt{8\sin^2\cos^2 t}.dt = 2\sqrt{2}\sin t\cos t.dt$$

$$S=\int dS = \int_0^{\pi/2} 2\pi\cos^2 t(2\sqrt{2}\sin t\cos t)dt = \int_0^{\pi/2} 4\sqrt{2}\pi\cos^3 t\sin t.dt$$

$$=-4\sqrt{2}\pi\left.\frac{\cos^4 t}{4}\right|_0^{\pi/2} = -\sqrt{2}\pi[\cos^4\frac{\pi}{2}-\cos^4 0] = -\sqrt{2}\pi[(0)-1]=\sqrt{2}\pi \text{ square units.}$$

syllabus

Homework:

1. Find the area of surfaces generated by revolving the curves indicated below about x-axis.

a. $y=\frac{x^3}{9}$ $\quad 0\le x\le 2$

b. $y=\sqrt{2x-x^2}$ $\quad 0\le x\le 2$

c. $y=\frac{x}{2}$ $\quad 0\le x\le 4$

d. $y=\sqrt{x+1}$ $\quad 1\le x\le 5$

2. Find the area of surfaces generated by revolving the curves indicated below about y-axis:

a. $x=\frac{y^3}{3}$ $\quad 0\le y\le 1$

b. $x=\sqrt{2y-1}$ $\quad \frac{1}{2}\le y\le 1$

c. $x=2y-1$ $\quad 1\le y\le 3$

3. Find the area of surfaces generated by revolving the curves indicated below about the stated axis:

a. $y=7x$, $0\le x\le 1$ about $y=2$.

b. $y=\sqrt{x}$, $1\le x\le 4$ about x-axis.

c. $y=\sqrt{4-x^2}$, $-1\le x\le 1$ about x-axis.

d. $x=\sqrt[3]{y}$, $1\le y\le 8$ about x-axis.

e. $x=9y+1$, $0\le y\le 2$ about $x=-1$.

f. $x=\sqrt{9-y^2}$, $-2\le y\le 2$ about y-axis.

g $x=y^3$, $0\le y\le 1$ about y-axis.

h. $x=2\sqrt{1-y}$, $-1\le y\le 0$ about y-axis.

i. $x=t^2$, $y=2t$, $0\le t\le 4$ about x-axis.

j. $x=r\cos t$, $y=r\sin t$, $0\le t\le \pi$ about x-axis.

k. $x=a\varphi-a\sin\varphi$, $y=a-a\cos\varphi$, $0\le\varphi\le 2\pi$ about x-axis.