. Length of Plane Curves:

i. Suppose that $y=f(x)$ is a smooth curve on the interval $[a, b]$, then:

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x_k)^2[1 + \frac{(\Delta y_k)^2}{(\Delta x_k)^2}]}$$

$$= \sqrt{[1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2]}.(\Delta x_k)$$

$$\therefore L = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{[1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2]}.(\Delta x_k)$$

When $n \to \infty \;\; \Rightarrow \;\; \Delta x \to 0$

So $\therefore L = \lim_{\Delta x_k \to 0} \sum_{k=1}^{\infty} \sqrt{[1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2]}.(\Delta x_k)$

Remember that $\lim_{\Delta x \to 0} \frac{\Delta y_k}{\Delta x_k} = f`(x)$

$$\therefore L = \int_a^b \sqrt{1 + [f`(x)]^2}\,dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{----(1)}$$

ii. Suppose that $x=f(y)$ is a continuous from $y=c$ to $y=d$, then the arc-length of the curve is:

$$L = \int_c^d \sqrt{1 + [f`(y)]^2}\,dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{----(2)}$$

iii. If the curve is represented by a parametric equations:

$x=x(t)$, $y=y(t)$ and $a \le t \le b$ and if $\frac{dx}{dt}$, $\frac{dy}{dt}$ are continuous functions on $a \le t \le b$, then the arc-length of the curve is:

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}.dt \qquad \text{----(3)}$$

Example 1: Find the length of the curve

$$y=\frac{4\sqrt{2}}{3}x^{3/2}-1; \qquad 0\le x\le 1.$$

Sol.: We use equation (1) with a=0 and b=1, and

$$y=\frac{4\sqrt{2}}{3}x^{3/2}-1$$

$$\frac{dy}{dx}=\frac{3}{2}*\frac{4\sqrt{2}}{3}x^{1/2}=2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2=\left(2\sqrt{2}x^{1/2}\right)^2=8x.$$

The length of the curve from x=0 to x=1 is

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\int_0^1\sqrt{1+8x}dx$$

$$=\frac{1}{8}\int_0^1\sqrt{1+8x}8.dx=\frac{1}{8}.\frac{(1+8x)^{3/2}}{3/2}\Bigg|_0^1$$

$$=\frac{1}{12}.[(1+8*1)^{3/2}-(1+8*0)^{3/2}]=\frac{13}{6} \text{ unit length.}$$

Example 2: Find the length of the curve $y=\left(\frac{x}{2}\right)^{2/3}$ from x=0 to x=2.

Sol.: The derivative:

$$\frac{dy}{dx}=\frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}*\frac{1}{2}=\frac{1}{3}\left(\frac{x}{2}\right)^{-1/3}$$

is not defined at x=0, so we can not find the curve's length with equation (1).

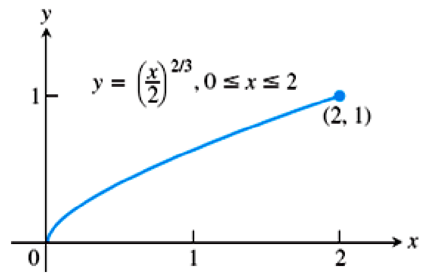
We therefore rewrite the equation to express x in term of y (x=$f(y)$):

$$y=\left(\frac{x}{2}\right)^{2/3} \Rightarrow y^{3/2}=\frac{x}{2} \Rightarrow x=2y^{3/2}$$

Note that when $x=0 \Rightarrow y=0$

and $x=1 \Rightarrow y=1$

from this we see that the curve whose length we want is also the graph $x=2y^{3/2}$ from $y=0$ to $y=1$



The derivative

$$\frac{dx}{dy}=2*\frac{3}{2}y^{1/2}=3y^{1/2}$$

is continuous from $y=0$ to $y=1$. We may therefore us equation (2) to find the curve's length:

$$L=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_0^1\sqrt{1+\left(3y^{1/2}\right)^2}\,dy$$

$$=\int_0^1\sqrt{1+9y}\,dy=\frac{1}{9}\frac{(1+9y)^{3/2}}{3/2}\Bigg|_0^1$$

$$=\frac{2}{27}[((1+9*1)^{3/2})-((1+9*0)^{3/2})]$$

$$=\frac{2}{27}(10\sqrt{10}-1)\approx 2.27 \text{ unit length.}$$

Example 3: Find the length of the circle of radius r defined parametrically by

$x=r\cos t$ and $y=r\sin t$ $\quad 0\le t\le 2\pi.$

Sol.: As the curve is defined by parametric equation, we use equation (3) to find the length of the curve

$$L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,.dt$$

We find $\quad \dfrac{dx}{dt}=-r\sin t \Rightarrow \left(\dfrac{dx}{dt}\right)^2=\left(-r\sin t\right)^2=r^2\sin^2 t$

$$\frac{dy}{dt} = r\cos t \quad \Rightarrow \quad \left(\frac{dy}{dt}\right)^2 = (r\cos t)^2 = r^2\cos^2 t$$

and $$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2\sin^2 t + r^2\cos^2 t$$

$$= r^2(\sin^2 t + \cos^2 t) = r^2.$$

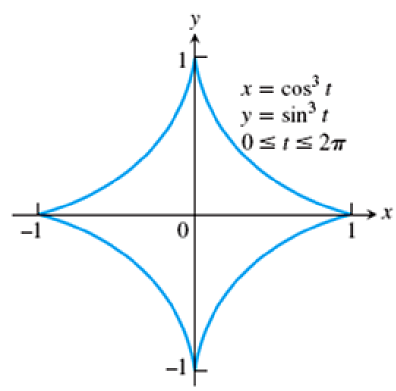
$$\therefore L = \int_0^{2\pi} \sqrt{r^2}.dt = \int_0^{2\pi} r.dt = r.t\Big|_0^{2\pi}$$

$= r(2\pi - 0) = 2\pi.r$ unit length.

Example 4: Find the length of the curve

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

Sol.: Because the curve's symmetry with respect to coordinate axes, its length is four times the length of the first quadrant portion. We have



$$x = \cos^3 t, \qquad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t.(-\sin t)]^2 = 9\cos^4 t\sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t.\cos t]^2 = 9\sin^4 t\cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}$$

$$= \sqrt{9\sin^2 t\cos^2 t} = |3\sin t\cos t|$$

$= 3\sin t\cos t$ (because $\sin t.\cos t \ge 0$ for $0 \le t \le \pi/2$)

Therefore: The Length of the first quadrant portion= $\int_0^{\pi/2} 3\cos t\sin t.dt$

$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t.dt = -\frac{3}{4}\cos 2t\Big|_0^{\pi/2} = \frac{3}{2}$$

syllabus

The length of the curve is four times this: $4(3/2) = 6$ unit length.

Homework: Find the length of the following curves:

1. $6xy = x^4 + 3$ from $x = 1$ to $x = 2$.
2. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$. (*Hint*: $1 + (dx/dy)^2$ is a perfect square.)
3. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$. (*Hint*: $1 + (dx/dy)^2$ is a perfect square.)
4. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$. (*Hint*: $1 + (dy/dx)^2$ is a perfect square.)
5. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$. (*Hint*: $1 + (dy/dx)^2$ is a perfect square.)
6. $x = \cos 2\theta$, $y = \sin 2\theta$ $0 \le \theta \le \pi/2$.
7. $x = t - \cos t$, $y = 1 + \sin t$ $-\pi \le t \le \pi$.