## Volumes by Cylindrical Shells (The strip is parallel to the axis of revolution)

The volume of the solid obtained by rotating the region bounded by $y=f(x)$ [where $f(x)>0$], $y=0$, $x=a$ and $x=b$ about the $y$-axis is obtained by:

Volume of typical cylindrical shell

$$dV=2\pi(r)(l)(t)$$

where $r$= radius of cylinder $= x$

$l$= cylindrical length (or height) $= f(x)$

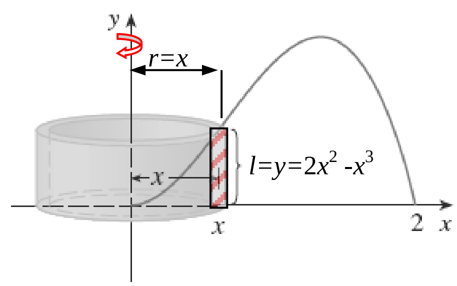
$t$= shell thickness $= dx$

$$\therefore dV = 2\pi.x.[f(x)]dx$$

and volume of the solid

$$V=\int dV=\int_a^b 2\pi[f(x)]x.dx$$

***Example 1:*** Find the volume of the solid obtained by rotating about the $y$-axis the region bounded by $y=2x^2-x^3$ and $y=0$.




**Sol.:** From the sketch of the curve we see that a typical shell has radius $x$ and height $y=f(x)$. So, by the shell method, the volume of typical shell is:

$$dV=2\pi r.l.t=2\pi x[f(x)]dx=2\pi x(2x^2-x^3)dx$$

To find the limits of integration put $y_{\text{curve}}=0$ so:

$$2x^2-x^3=0 \quad \Rightarrow \quad x^2(2-x)=0$$

$\therefore$ either $x^2=0 \quad \Rightarrow \quad x=0$

or $(2-x)=0 \quad \Rightarrow \quad x=2$

so the volume of the solid:

$$V=\int dV=\int_0^2 2\pi(2x^3-x^4)dx=2\pi[2*\frac{x^4}{4}-\frac{x^5}{5}]\Big|_0^2$$

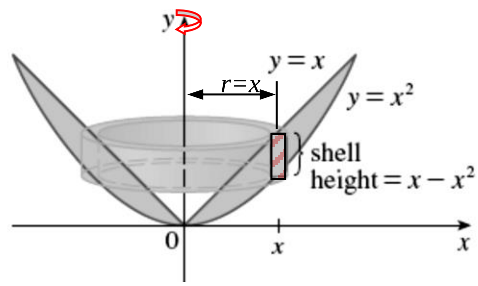
$$=2\pi[(2*\frac{2^4}{2}-\frac{2^5}{5})-(0)]=2\pi[16-\frac{32}{5}]=\frac{96}{5}\pi \text{ cubic units}$$

**Example 2:** Find the volume of the solid obtained by rotating about the *y*-axis the region between $y=x$ and $y=x^2$.

**Sol.:** when we sketch the region we see that the shell has radius $x$, and height $x$-$x^2$.

So the volume of typical cylindrical shell:

$$dV=2\pi r.l.t=2\pi x(x-x^2)dx$$




To find the limits of integration put $y_{curve}=y_{line}$ so

$$x=x^2 \quad\Rightarrow\quad x-x^2=0 \quad\Rightarrow\quad x(1-x)=0$$

$\therefore$ either $x=0 \quad\Rightarrow\quad y=0$

or $(1-x)=0 \quad\Rightarrow\quad x=1$ and $y=1$

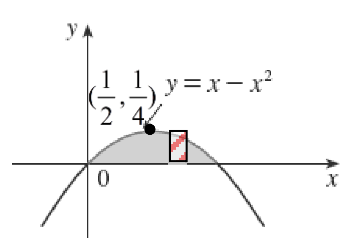
So the volume of the solid:

$$V=\int dV=\int_0^1 2\pi(x^2-x^3)dx=2\pi[\frac{x^3}{3}-\frac{x^4}{4}]\Big|_0^1$$

$$=2\pi[(\frac{1^3}{3}-\frac{1^4}{4})-(0)]=2\pi[\frac{1}{3}-\frac{1}{4}]=\frac{\pi}{6} \text{ cubic units}$$

**Example 3:** Find the volume of the solid obtained by rotating the region bounded by $y=x-x^2$ and $y=0$ about the line $x=2$.

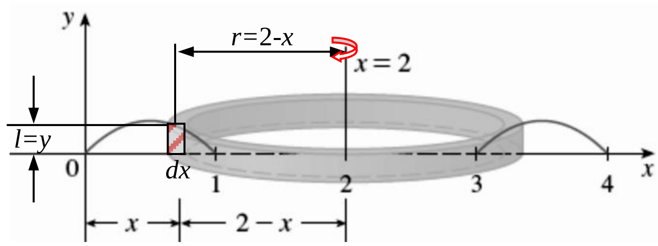
**Sol.:** To graph the curve $y=x-x^2$, complete the square and compare the resulting equation with the curve $y=-x^2$




$$y=-(x^2-x+\frac{1}{4})+\frac{1}{4}=-(x-\frac{1}{2})^2+\frac{1}{4}$$

Sketch the region by shifting the curve $y=-x^2$ by 1/2 units left and 1/4 units up.

The volume of typical cylindrical shell:




$$dV = 2\pi r.l.t$$

Where $r$=2-$x$, $l=y=x-x^2$ and $t=dx$

$$\therefore dV = 2\pi(2-x)(x-x^2).dx$$

$$= 2\pi(2x-2x^2-x^2+x^3)dx$$

$$= 2\pi(2x-3x^2+x^3)dx$$

To find the limits of integration put $y_{\text{curve}} = 0 \Rightarrow x-x^2 = 0 \quad \Rightarrow x(1-x) = 0$

$\therefore$ either $x=0$

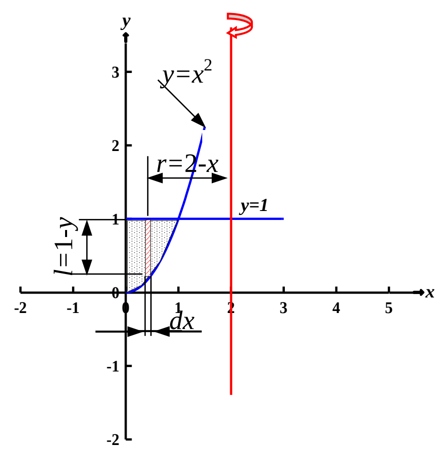
or $(1-x)=0 \quad \Rightarrow \quad x=1$

So the volume of the solid:

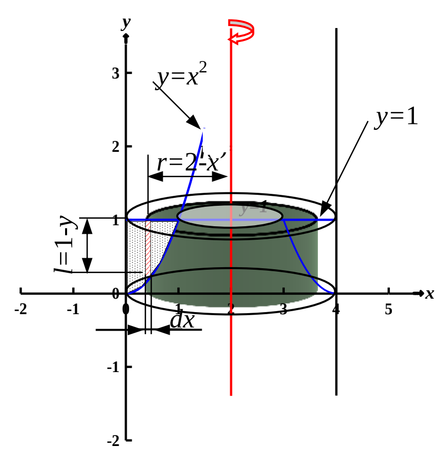
$$\therefore V = \int dV = \int_0^1 2\pi(2x-3x^2+x^3).dx$$

$$= 2\pi(x^2-x^3+\frac{x^4}{4})\Big|_0^1$$

$$= 2\pi[(1^2-1^3+\frac{1^4}{4})-(0)] = \frac{\pi}{2}$$




**Example 4:** The region bounded by the parabola $y=x^2$, the $y$-axis and the line $y$=1in the first quadrant is revolved about the line $x$=2 to generate a solid. Find the volume of the solid.




**Sol.:** $l$=1-$y$, $r$=2-$x$ and $t=dx$

$dV=2\pi.r.l.t=2\pi(2-x)(1-y)dx$

$=2\pi(2-x)(1-x^2)dx$

The limits of integration from $x$=0 to $x$=1.

The volume of the solid:

$$\therefore V = \int dV = \int_0^1 2\pi(2-x)(1-x^2)dx$$

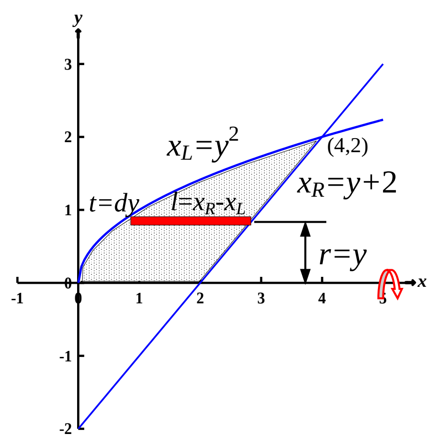
$$= \int_0^1 2\pi(2-2x^2-x+x^3)dx = 2\pi(2x - \frac{2x^3}{3} - \frac{x^2}{2} + \frac{x^4}{4})\Big|_0^1$$

$$= 2\pi[(2*1 - \frac{2*1^3}{3} - \frac{1^2}{2} + \frac{1^4}{4}) - (0)]$$

$$= \frac{13}{6}\pi \text{ cubic units}$$

***Example 5:*** Find the volume of the solid which is generated by rotating the region bounded by $y = \sqrt{x}$ , $y$=$x$-2 and $x$-axis about:

a. $x$-axis. b. $y$-axis.




**Sol.:** a. about $x$-axis (the strip is parallel to the axis of rotation so it will give cylindrical shell)

$dV$=2$\pi$.$r$.$l$.$t$

where $r$=$y$, $l$=$x_R$-$x_L$=($y$+2)-($y^2$)= $y$+2-$y^2$ and $t$=$dy$

$$\therefore dV = 2\pi.y(y+2-y^2)dy = 2\pi(2y+y^2-y^3)dy$$

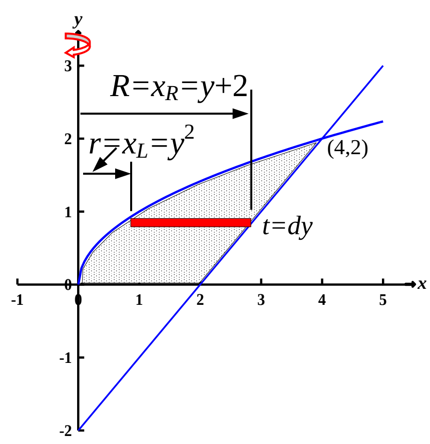
The limits of integration from $y$=0 to $y$=2

So the volume of the solid:

$$V = \int dV = \int_0^2 2\pi(2y+y^2-y^3)dy$$

$$= 2\pi(y^2 + \frac{y^3}{3} - \frac{y^4}{4})\Big|_0^2$$

$$= 2\pi[(2^2 + \frac{2^3}{3} - \frac{2^4}{4}) - (0)] = \frac{16}{3}\pi$$ cubic units



b. about $y$-axis (the strip is perpendicular to the axis of rotation so it will give washer)

$$dV=\pi(R^2-r^2).t$$

where $R=x_R=y+2$, $r=x_L=y^2$ and $t=dy$

$$\therefore dV=\pi.[(y+2)^2-(y^2)^2]dy$$

$$=\pi[y^2+4y+4-y^4]dy$$

The limits of integration from $y=0$ to $y=2$

So the volume of the solid:

$$V=\int dV=\int_0^2 \pi(y^2+4y+4-y^4)dy$$

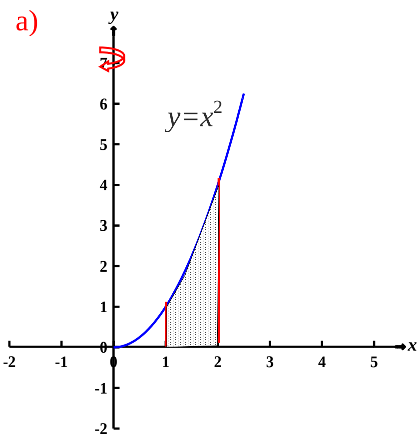
$$=\pi\left(\frac{y^3}{3}+2y^2+4y-\frac{y^5}{5}\right)\Big|_0^2$$

$$=\pi\left[\left(\frac{2^3}{3}+2*2^2+4*2-\frac{2^5}{5}\right)-(0)\right]=\frac{184}{15}\pi \text{ cubic units}$$

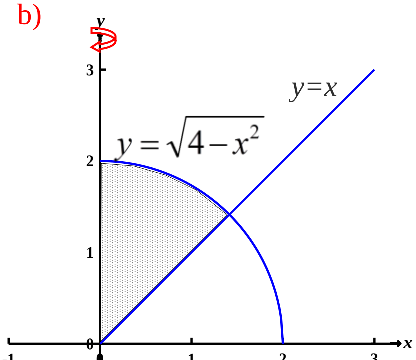
## Homework

1. Use the cylindrical shells to find the volume of the solids generated when the shaded area is revolved about the indicated axis.

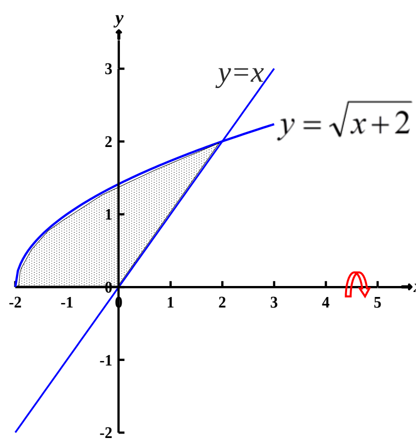
a)



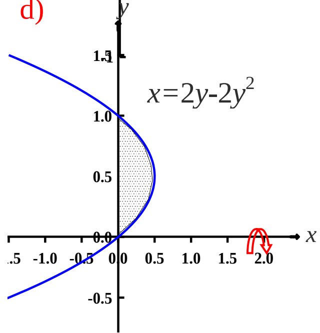

b)




c)




d)



2. Use cylinderical shells to find the volume of the solids generated when the region enclosed by the given curve is revolved about the given axis.
   a. $y=x^3$, $x=1$ and $y=0$ about $y$-axis.
   b. $y=\sqrt{x}$, $x=0$, $x=9$ and $y=0$ about $y$-axis.
   c. $y=\cos x^2$, $x=0$, $x=\sqrt{\frac{\pi}{2}}$ and $y=0$ about $y$-axis.
   d. $y=2x-x^2$ and $y=0$ about $y$-axis.
   e. $y^2=x$, $y=1$ and $x=0$ about $x$-axis.
   f. $x=2y$, $y=2$, $y=3$ and $x=0$ about $x$-axis.
   g. $y=x^2$, $x=1$ and $y=0$ about $x$-axis.
   h. $y=\frac{1}{x^3}$, $x=1$, $x=2$ and $y=0$ about $x=-1$.
   i. $y=x^3$, $y=1$ and $x=0$ about $y=1$.
3. Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines about $y$-axis for:
   a. $y=x$, $y=-x/2$ and $x=2$.
   b. $y=x^2+1$, $y=0$, $x=0$ and $x=1$.
4. As in *problem 3* but rotate about $x$-axis for the following:
   a. $y=|x|$ and $y=1$.
   b. The parabola $x=2y-y^2$ and $y$-axis.

*syllabus*

*Mathematics*