Volume of Solids of Revolution:

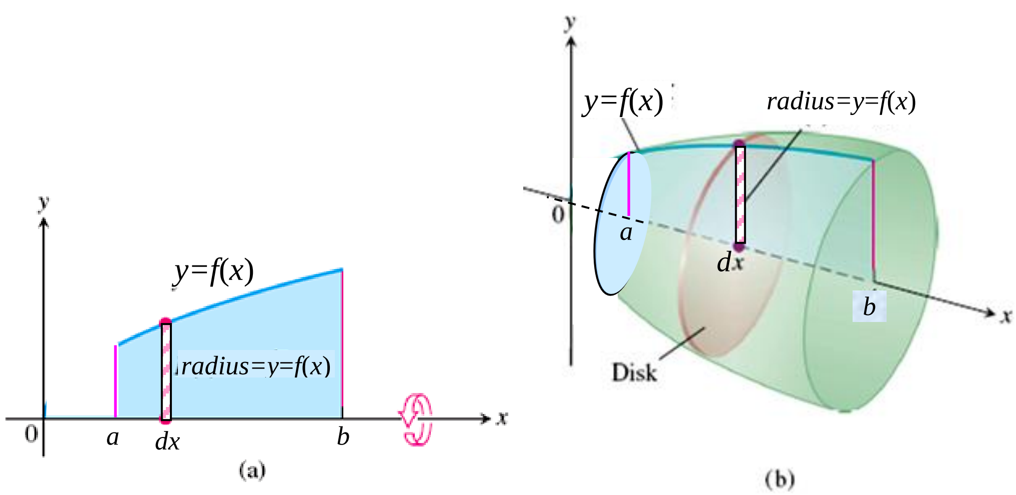
The Solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution**. We will use the following methods to find this volume

a. The Disk Method (The strip is perpendicular to the axis of revolution):

i. Rotation about x-axis: The volume of the solid generated by revolving the region between the graph of continuous function $y=f(x)$ and the x-axis from $x=a$ to $x=b$ about the x-axis is

$$dV = \pi.(radius)^2(thickness) = \pi.y^2dx = \pi.\left(f(x)\right)^2dx$$

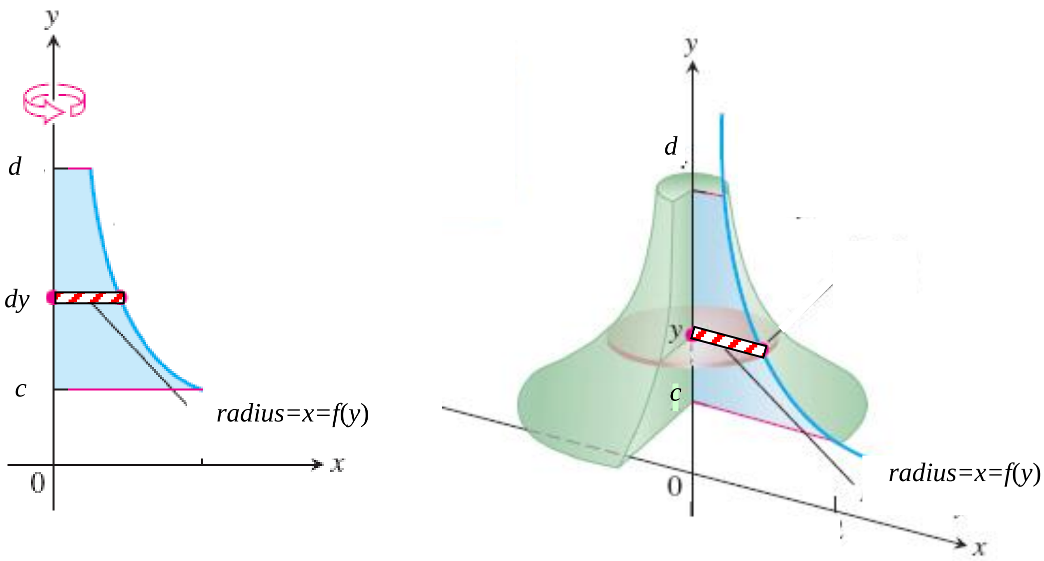
$$\text{Volume} = \int dV = \int_a^b \pi(radius)^2dx = \int_a^b \pi\left(f(x)\right)^2dx$$



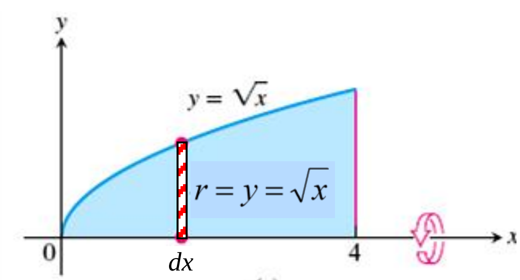
ii. Rotation about y-axis: If the region bounded between the continuous function $x=f(y)$ and y-axis is rotated about y-axis from $y=c$ to $y=d$ to generate a solid, then the volume of the solid is:

$$dV = \pi.(radius)^2(thickness) = \pi.x^2dy = \pi.\left(f(y)\right)^2dy$$

$$\text{Volume} = \int dV = \int_c^d \pi(radius)^2dy = \int_c^d \pi\left(f(y)\right)^2dy.$$



Example 1: The region between the curve $y=\sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.



(a)

Sol.: We draw figures showing the region, the typical radius and the generated solid. The volume of the disk is

$$dV=\pi.(radius)^2(thickness)=\pi.r^2.t$$

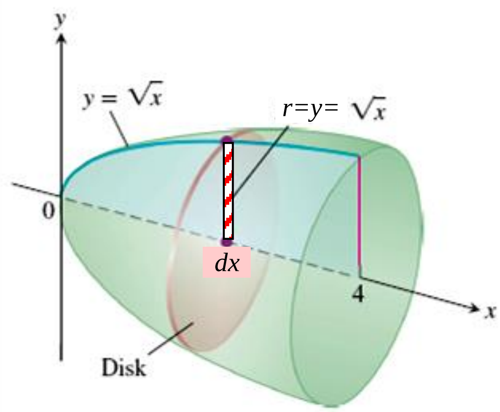
Where $r=y=f(x)=\sqrt{x}$ and $t=dx$

$$\therefore dV=\pi(\sqrt{x})^2 dx=\pi.x.dx$$

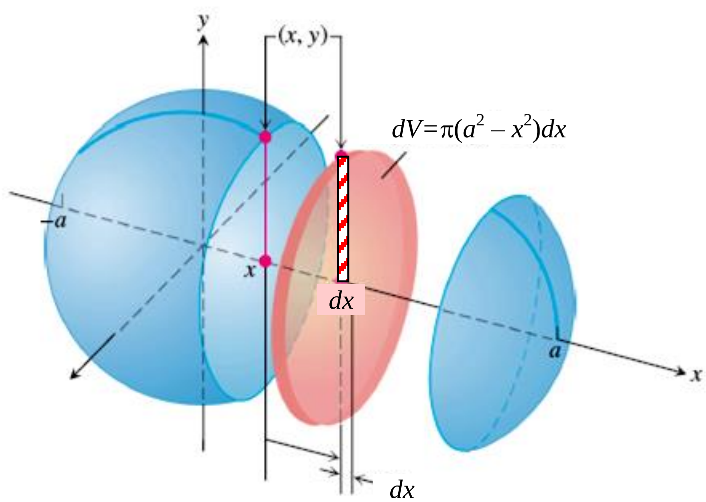
So the volume of the solid is

$$V=\int dV=\int_0^4 \pi.x.dx=\pi\frac{x^2}{2}\Big|_0^4$$

$$=\frac{\pi}{2}[4^2-0^2]=\frac{16\pi}{2}=8\pi \text{ cubic units}$$



Example 2: The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume.



Sol.: We imagine a sphere cut into thin slices by planes perpendicular to the x-axis. The volume of a typical slice at point x between a and $-a$ is

$$dV = \pi . r^2 . t = \pi . y^2 dx = \pi (a^2 - x^2) dx$$

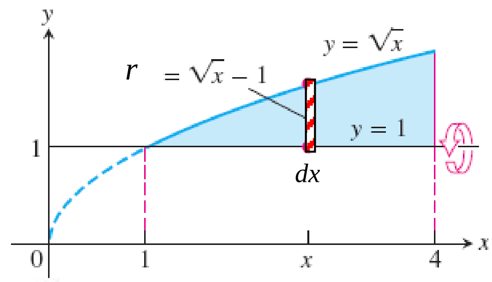
Therefore the volume is

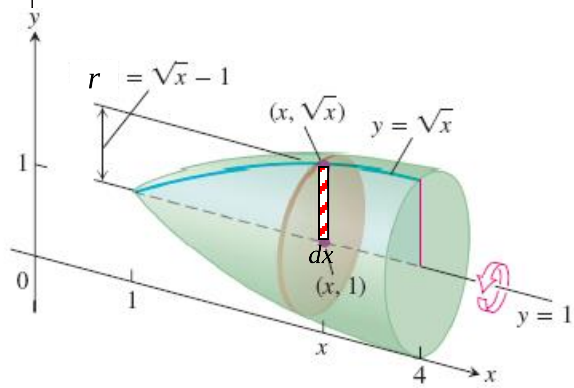
$$V = \int dV = \int_{-a}^{a} \pi (a^2 - x^2) dx = 2\int_{0}^{a} \pi (a^2 - x^2) dx$$

$$= 2\pi (a^2 x - \frac{x^3}{3}) \Big|_0^a = \frac{4}{3}\pi . a^3$$

Example 3: Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$.

Sol.: We draw figures showing the region, the typical radius and the generated solid. The volume of the disk is





$$dV = \pi.(radius)^2(thickness) = \pi.r^2.t$$

Where $r = y - 1 = \sqrt{x} - 1$ and $t = dx$

$$\therefore dV = \pi(\sqrt{x} - 1)^2 dx$$

So the volume of the solid is

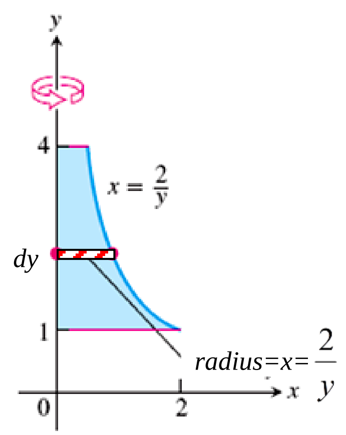
$$V = \int dV = \int_1^4 \pi(\sqrt{x} - 1)^2 dx = \int_1^4 \pi(x - 2\sqrt{x} + 1$$

$$= \pi(\frac{x^2}{2} - \frac{2x^{3/2}}{3/2} + x)\Big|_1^4$$

$$= \pi[(\frac{4^2}{2} - \frac{4 * 4^{3/2}}{3} + 4) - (\frac{1^2}{2} - \frac{4 * 1^{3/2}}{3} + 1)]$$

$$= \frac{7\pi}{6} \text{ cubic units}$$

Example 4: Find the volume of the solid generated by revolving the region between the *y*-axis and the curve $x = 2/y$, $1 \le y \le 4$, about *y*-axis.



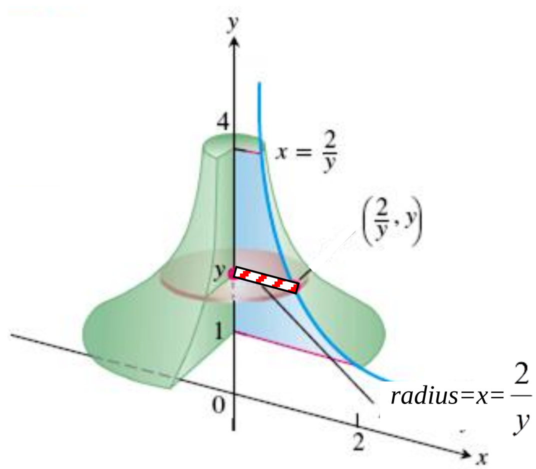
Sol.: We draw figures showing the region, the typical radius and the generated solid. The volume of the disk is

$$dV = \pi.(radius)^2(thickness) = \pi.r^2.t$$

Where $r = x = \dfrac{2}{y}$ and $t = dy$

$$\therefore dV = \pi(\frac{2}{y})^2 dy = \frac{4\pi}{y^2} dy$$

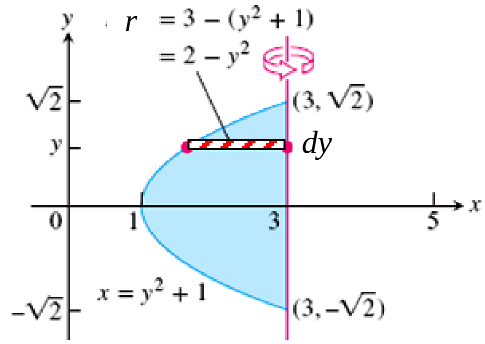
So the volume of the solid is



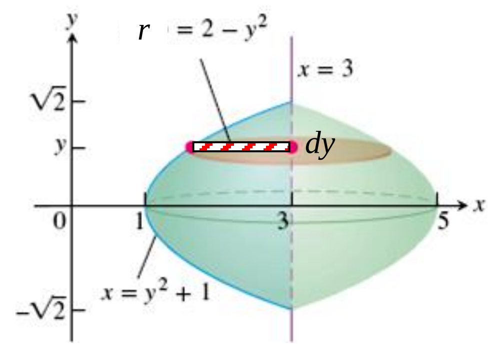
$$V = \int dV = \int_1^4 \frac{4\pi}{y^2}\,dy = -\frac{4\pi}{y}\Big|_1^4$$

$$= 4\pi[-\frac{1}{4} - (-\frac{1}{1})] = 4\pi * \frac{3}{4}$$

$$= 3\pi \text{ cubic units}$$



Example 5: Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$, about $x = 3$.



Sol.: We draw figures showing the region, the typical radius and the generated solid. Note that the cross-sections are perpendicular to the line $x = 3$. The volume of the disk is

$$dV = \pi.(radius)^2(thickness) = \pi.r^2.t$$

Where $r = 3 - x = 3 - (y^2 + 1) = 2 - y^2$ and $t = dy$

$$\therefore dV = \pi(2 - y^2)^2 dy$$

So the volume of the solid is

$$V = \int dV = \int_{-\sqrt{2}}^{\sqrt{2}} \pi(2 - y^2)^2 dy$$

$$= \int_{\sqrt{2}}^{\sqrt{2}} \pi(4 - 4y^2 + y^4)dy = 2\int_0^{\sqrt{2}} \pi(4 - 4y^2 + y^4)dy \qquad = 2\pi(4y - \frac{4}{3}y^3 + \frac{y^5}{5})\Big|_0^{\sqrt{2}}$$

$$= 2\pi[(4\sqrt{2} - \frac{4}{3}(\sqrt{2})^3 + \frac{(\sqrt{2})^5}{5}) - (0)] = \frac{64\pi\sqrt{2}}{15} \text{ cubic units}$$

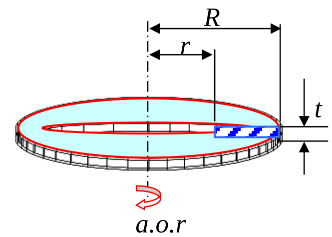
b. The Washer Method (The strip is perpendicular to the axis of revolution):

If the region we revolved to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it. The cross-sections perpendicular to the axis of revolution are washers instead of disks. The dimensions of a typical washer are

Outer radius: R

Inner radius: r

Thickness: t



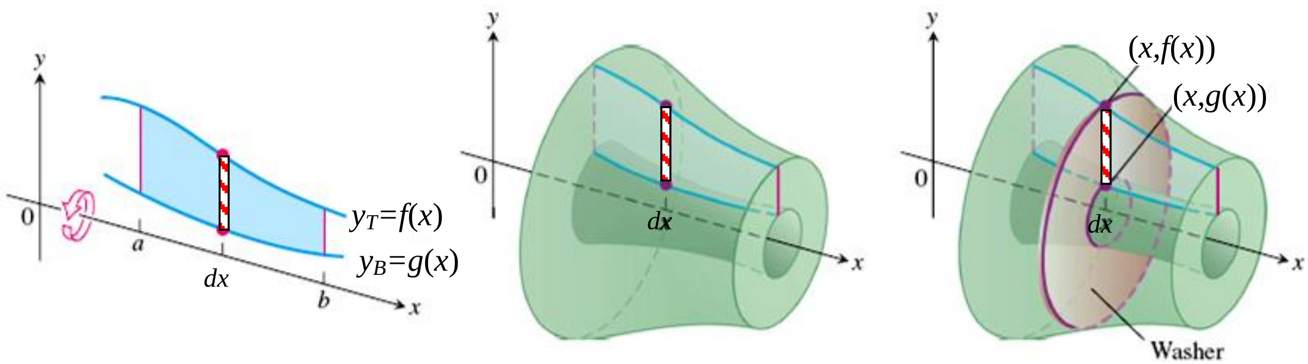
The washer's volume is: $dV = \pi\,[R^2 - r^2]\,t$

i. Rotation about *x*-axis:

If a region bounded by curves with equations $y=f(x)$, $y=g(x)$, $x=a$, and $x=b$, where f and g are continuous and $f(x) \geq g(x)$ for $a \leq y \leq b$ is rotated about x-axis then,

$R=y_T=f(x)$, $r=y_B=g(x)$ and $t=dx$

$$V = \int dV = \int_a^b \pi\{R^2 - r^2\}dx = \int_a^b \pi\{(y_T)^2 - (y_B)^2\}dx = \int_a^b \pi\{[f(x)]^2 - [g(x)]^2\}$$



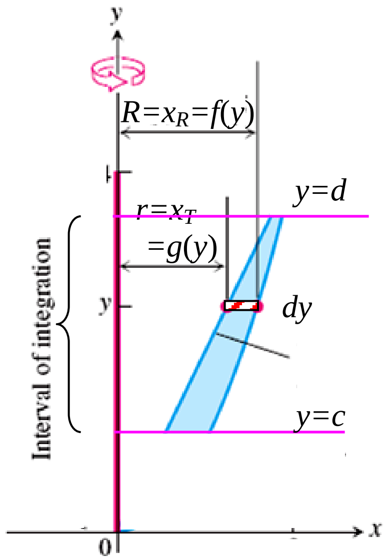
ii. Rotation about *y*-axis:

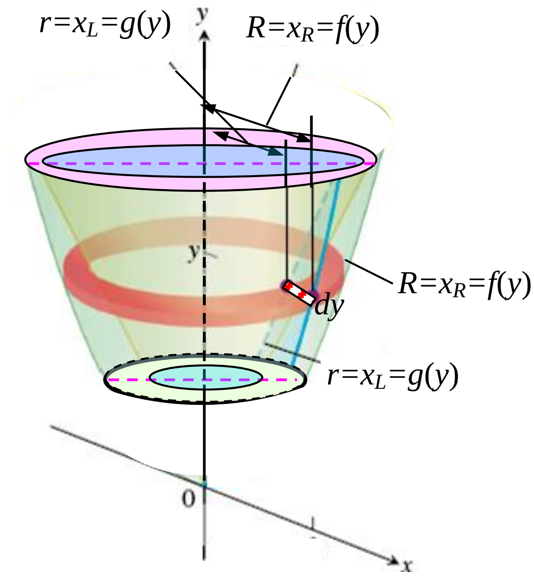
If a region bounded by curves with equations $x=f(y)$, $x=g(y)$, $y=c$, and $y=d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ is rotated about y-axis then,

$R=x_R=f(y)$, $r=x_L=g(y)$ and $t=dy$;

and the volume of the solid:

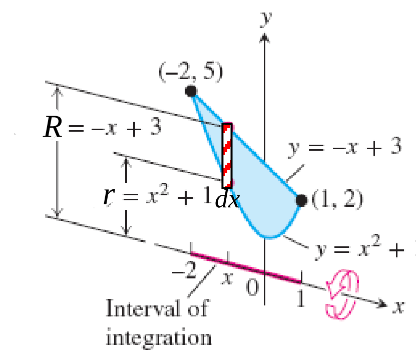
$$V = \int dV = \int_c^d \pi\{R^2 - r^2\}dy = \int_c^d \pi\{(x_R)^2 - (x_L)^2\}dy = \int_c^d \pi\{[f(y)]^2 - [g(y)]^2\}dy$$





Example 6: The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find its volume.

Sol.:



1. Draw the region and sketch a strip across it perpendicular to the axis of revolution.
2. Find the outer and the inner radii of the washer that would be swept out by the strip if it were revolved about the x-axis along with the region.

   These radii are the distance of the ends of the strip from the axis of revolution.

   Outer radius: $R = y_T = -x + 3$

   Inner radius: $r = y_B = x^2 + 1$

3. Find the limits of integration by finding the x-coordinate of the intersection points of the curve and line (put $y_{\text{curve}} = y_{\text{line}}$).

$$x^2 + 1 = -x + 3$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$

Either $\quad x = -2 \quad \Rightarrow \quad y = 5$

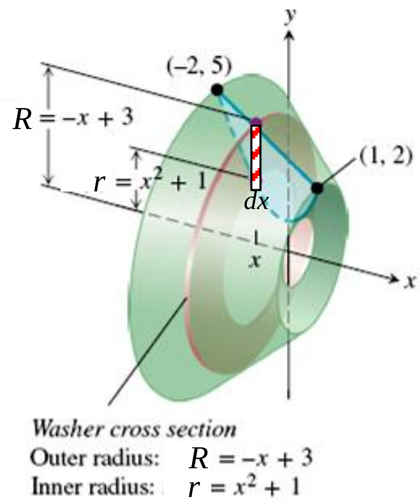
or $\quad x = 1 \quad \Rightarrow \quad y = 2$

4. Find the washer's volume

$$dV = \pi\,[R^2 - r^2]\,t$$

where $R = y_T = -x + 3$, $r = y_B = x^2 + 1$ and $t = dx$

$$\therefore dV = \pi\,[(-x + 3)^2 - (x^2 + 1)^2]\,dx$$
$$= \pi\,[x^2 - 6x + 9 - (x^4 + 2x^2 + 1)]\,dx$$
$$= \pi\,[\,8 - 6x - x^2 - x^4\,]\,dx$$



5. Evaluate the volume integral

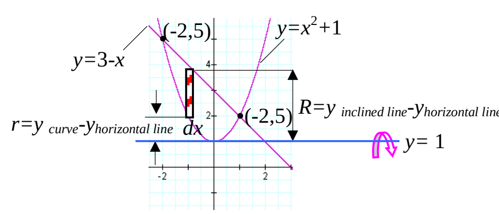
$$V = \int dV = \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)dx$$

$$= \pi(8x - \frac{6x^2}{2} - \frac{x^3}{3} - \frac{x^5}{5})\Big|_{-2}^{1} = \frac{117\pi}{5} \text{ cubic units.}$$

Example 7: Repeat Example 6 but here rotate about the line $y=1$

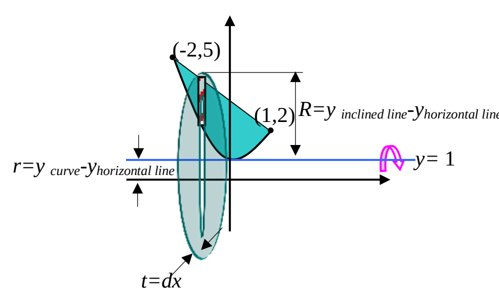
Sol.:

1. Draw the region and sketch a strip across it perpendicular to the axis of revolution.



2. Find the outer and the inner radii of the washer that would be swept out by the strip. These radii are the distance of the ends of the strip from the axis of revolution.

Outer radius: $R = y_{\text{inclined line}} - y_{\text{horizontal line}}$



$=(-x+3)-(1)=-x+3-1=-x+2$

Inner radius: $r = y_{curve} - y_{\text{horizontal line}}$

$=(x^2+1)-(1)=x^2$

3. Find the limits of integration: from previous example the limit of integration are from $x = -2$ to $x = 1$
4. Find the washer's volume

$dV = \pi\,[R^2 - r^2]\,t$

where $R = 2 - x$, $r = x^2$ and $t = dx$

$\therefore dV = \pi\,[(2-x)^2 - (x^2)^2]\,dx$

$= \pi\,[4 - 4x - x^2 - x^4)]\,dx$

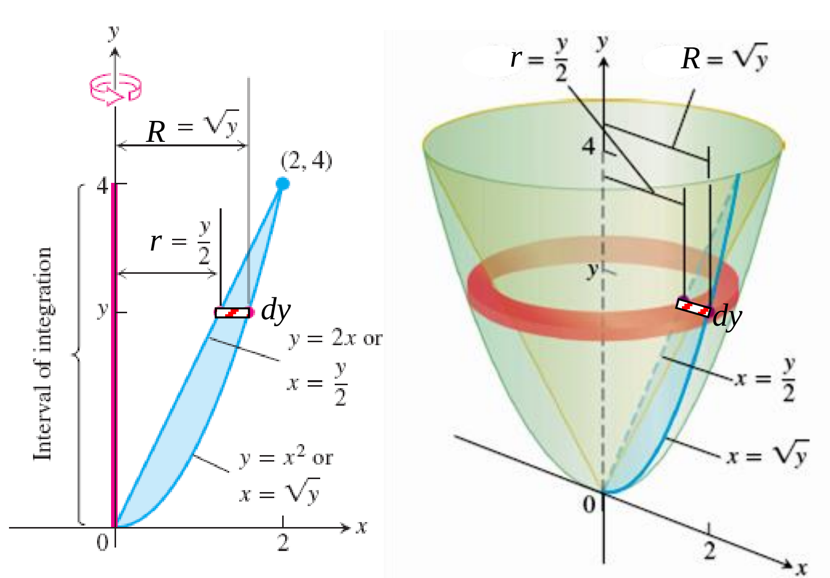
5. Evaluate the volume integral

$$V = \int dV = \int_{-2}^{1} \pi(4 - 4x - x^2 - x^4)dx$$

$$= \pi(4x - \frac{4x^2}{2} - \frac{x^3}{3} - \frac{x^5}{5})\Big|_{-2}^{1} = \frac{75\pi}{5} \text{ cubic units.}$$

Example 8: The region bounded by the parabola $y = x^2$ and the line $y = 2x$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Sol.: First we draw the region and draw a strip across it perpendicular to the axis of revolution (the y-axis). The radii



of washer swept out by the strip are $R = x_R = \sqrt{y}$ and $r = x_L = y/2$ but its thickness is $t=dy$

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$dV = \pi[R^2 - r^2]t = \pi[(\sqrt{y})^2 - (y/2)^2]dy = \pi[y - \frac{y^2}{4}]dy$$

$$V = \int dV = \int_0^4 \pi[y - \frac{y^2}{4}]dy = \pi[\frac{y^2}{2} - \frac{y^3}{3*4}]\Bigg|_0^4 = \frac{8}{3}\pi \text{ cubic units}$$

Homework:

1. use the disk method to find the volumes of the solids generated by revolving the given lines and curves about the given axis of rotation (*a.o.r*)
   a. The region bounded by: $y = \sqrt{9 - x^2}$ and $y=0$ about x-axis.
   b. The region bounded by: $y = x^3$, $y=0$ and $x=2$ about x-axis.
   c. The region bounded by: $y = \sqrt{\cos x}$, $0 \le x \le \frac{\pi}{2}$, $y=0$ and $x=0$ about x-axis.
   d. The region bounded by: $y = \sec x$, $y=0$ $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$ about x-axis.
   e. The region bounded by: $x = \sqrt{4 - y}$, $x=0$, and $y=0$ about y-axis.
   f. The region bounded by: $x = 1 - y^2$ and $x=0$ about y-axis.
   g. The region bounded by: $y = \frac{2}{\sqrt{x+1}}$, $x=0$, $y=0$ and $x=3$ about y-axis.
2. Use the washer method to find the volume of the solids generated by revolving about the given axis of rotation the regions bounded by the lines and curves in the following exercises.
   a. The region bounded by: $y = x^2 + 3$ and $y=4$ about x-axis.
   b. The region bounded by: $y = x^2 + 3$ and $y=x+3$ about x-axis.

c. The region bounded by: $y = \sec x$, $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ and $y = \sqrt{2}$ about *x*-axis.

d. The region bounded by: $y = \sqrt{x}$, and $y=x$ about *y*-axis.

e. The region bounded by: $y = x^2$, $y=0$ and $x=2$ about *y*-axis.

f. The region bounded by: $y = \sqrt{x}$, $y=2$ and $x=0$ about $y=4$.

g. The region bounded by: $y = \sqrt{x}$, $y=2$ and $x=0$ about $x=4$.

h. The region bounded by: $y = \sin x$, $0 \le x \le \pi$ and $y = \frac{1}{\sqrt{2}}$ about $y = \frac{1}{\sqrt{2}}$.

[syllabus]

Mathematics