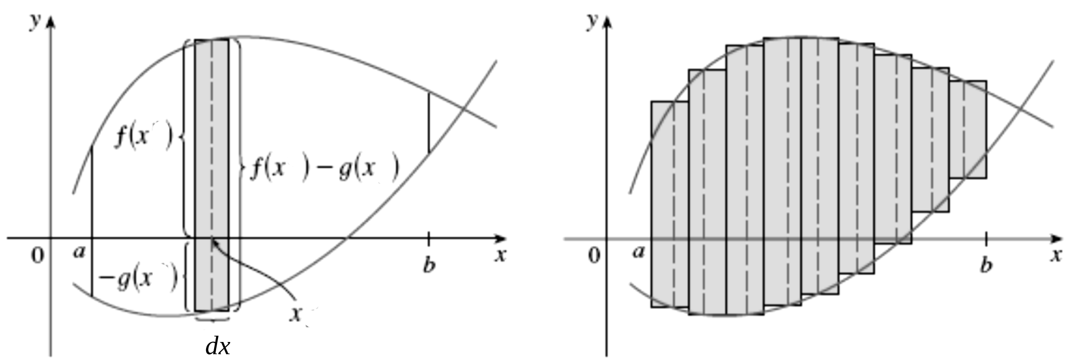
APPLICATIONS OF DEFINITE INTEGRAL

1. Area between Curves:

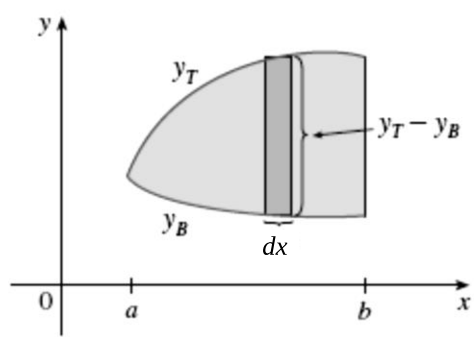
The area A of the region bounded by the curves $y=f(x)$, $y=g(x)$ and the lines $x=a$, $x=b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A=\int_a^b [f(x)-g(x)]dx$$



(a) Typical rectangle (b) Approximating rectangles

Steps to find area between two curves:



1. Sketch the graph of the curves together. This identify the up curve y_T and the bottom curve y_B
2. Find the limits of integration (if not given in the problem).
3. Write a formula of $[f(x) - g(x)]$ or $[y_T\text{-}y_B]$ and simplify it.
4. Integrate $[f(x) - g(x)]$ from a to b. The number you get it is the area.

Example 1: Find the area of the region enclosed by the parabolas $y = x^2$ and

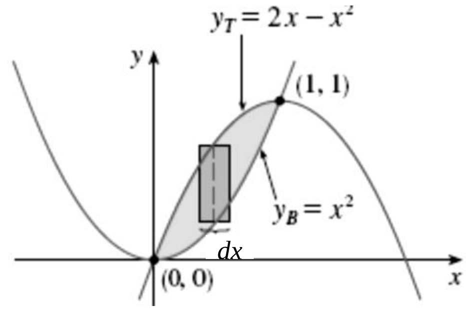
$y = 2x - x^2$

Sol.: We first find the points of intersection of the parabolas by solving their equations simultaneously.

$$x^2 = 2x - x^2 \;\Rightarrow\; x^2 + x^2 - 2x = 0 \;\Rightarrow\; 2x^2 - 2x = 0 \;\Rightarrow\; 2x(x-1) = 0$$

either $2x = 0 \;\Rightarrow\; x = 0 \;\Rightarrow\; y = 0$

or $x - 1 = 0 \;\Rightarrow\; x = 1 \;\Rightarrow\; y = 1$



The points of intersection are (0,0) and (1,1)

We see from Figure that the top and bottom boundaries are

$$y_T = 2x - x^2 \text{ and } y_B = x^2$$

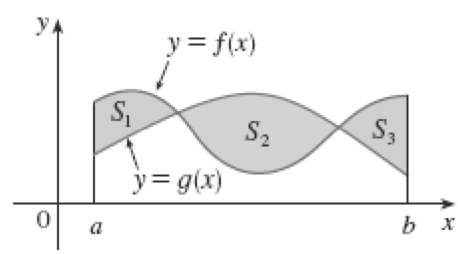
The area of a typical rectangle is

$$dA = y_T - y_B = (2x - x^2) - (x^2) = 2x - x^2 - x^2 = 2x - 2x^2$$

and the region lies between x=0 and x= 1. So the total area is

$$A = \int dA = \int_0^1 (2x - 2x^2)dx = \frac{2x^2}{2} - \frac{2x^3}{3}\Big|_0^1 = [(1)^2 - \frac{2(1)^3}{3}] - [0] = \frac{1}{3} \text{ square units}$$

If we are asked to find the area between the curves $y=f(x)$ and $y=g(x)$ where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for values of x, then we split the given region S into several regions $S_1, S_2, \ldots$ with areas $A_1, A_2, \ldots$ as shown in Figure. We then define the area of the region S to be the sum of the areas of the smaller regions $S_1, S_2, \ldots$ that is, $A=A_1+A_2+\ldots$. Since

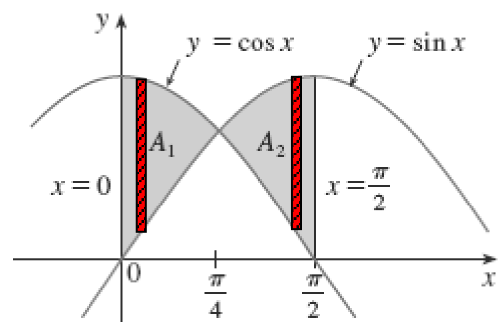


$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & when \quad f(x) \geq g(x) \\ g(x) - f(x) & when \quad g(x) \geq f(x) \end{cases}$$

Example 2: Find the area of the region bounded by the curves $y=\sin x$, $y=\cos x$, $x=0$, and $x=\pi/2$.

Sol.: The point of intersection occur when $\sin x = \cos x$, that is, when $x=\pi/4$.

Observe that $\cos x \geq \sin x$ when $0 \leq x \leq \pi/4$ but $\sin x \geq \cos x$ when $\pi/4 \leq x \leq \pi/2$. Therefore the required area is



$$A = \int_0^{\pi/2} |\cos x - \sin x| dx = A_1 + A_2$$

$$= \int_0^{\pi/4} [\cos x - \sin x] dx + \int_{\pi/4}^{\pi/2} [\sin x - \cos x] dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

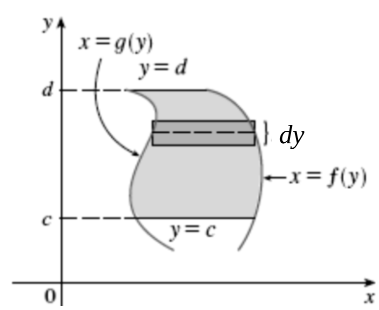
$$= 2\sqrt{2} - 2$$

In this particular example we could have saved some work by noticing that the region is symmetric about $x=\pi/4$ and so,

$$A = 2A_1 = 2\int_0^{\pi/4} [\cos x - \sin x] dx$$

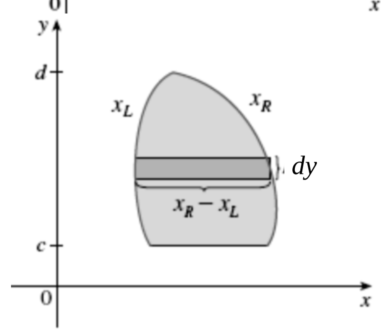
Integration with respect to *y* (horizontal strip)

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x=f(y)$, $x=g(y)$, $y=c$, and $y=d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ then its area is



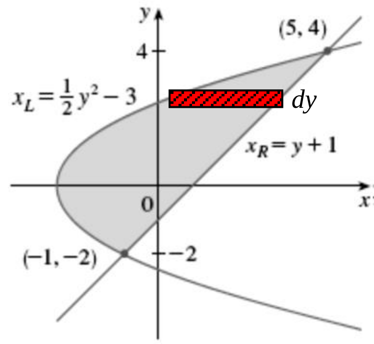
$$A = \int_c^d [f(y) - g(y)] dy$$

If we write for the right boundary x_R and for the left boundary x_L, then we have



$$A = \int_c^d [x_R - x_L] dy$$

Example 3: Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$



Sol.: To find points of intersections put $x_{\text{line}} = x_{\text{curve}}$ so

$$y + 1 = \frac{y^2 - 6}{2} \Rightarrow 2(y+1) = y^2 - 6 \Rightarrow y^2 - 2y - 8 = 0$$

$$\Rightarrow (y-4)(y+2) = 0 \text{ either } y=4 \Rightarrow x=5$$

$$\text{or} \quad y=-2 \Rightarrow x=-1$$

$\therefore$ (5,4) and (-1,-2) are the points of intersections of the two curves.

We can notice from Figure that the left and right boundary curves are

$$x_R = y + 1 \quad \text{and} \quad x_L = \frac{1}{2}y^2 - 3$$

We must integrate between the appropriate y-values, y=-2 and y=4. Thus

$$A = \int_{-2}^{4} [x_R - x_L] dy$$

$$= \int_{-2}^{4} [(y+1) - (\frac{1}{2}y^2 - 3)] dy$$

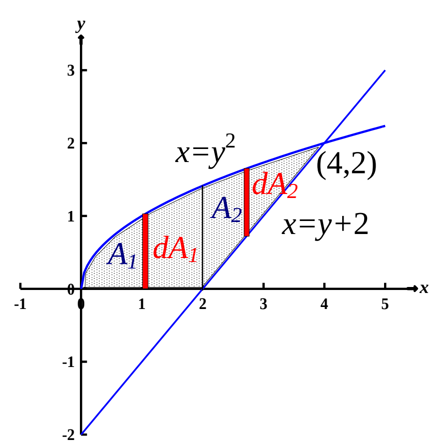
$$= \int_{-2}^{4} [-\frac{1}{2}y^2 + y + 4)] dy$$

$$= [-\frac{y^3}{2*3} + \frac{y^2}{2} + 4y]_{-2}^{4}$$

$$= \left(-\frac{4^3}{6} + \frac{4^2}{2} + 4*4\right) - \left(-\frac{(-2)^3}{6} + \frac{(-2)^2}{2} + 4*(-2)\right)$$

$$-\frac{64}{6} + 8 + 16 - \frac{8}{6} - 2 + 8 = 18 \text{ square units.}$$

Example 4: Find the area of the region between the curves $x=y^2$ and $x=y+2$ in the first quadrant.



Sol.: Graph the curves together

a. **Using vertical strip: we should split the are into two areas by the line *x*=2**

$$\therefore A = A_1 + A_2$$

The area of the first typical rectangle

$$dA_1 = (y_T - 0)dx = (\sqrt{x} - 0)dx = \sqrt{x}dx$$

$$\therefore A_1 = \int dA_1 = \int_0^2 \sqrt{x}dx = \left.\frac{x^{3/2}}{3/2}\right|_0^2 = \frac{2}{3}[2^{3/2} - 0] = 1.885618$$

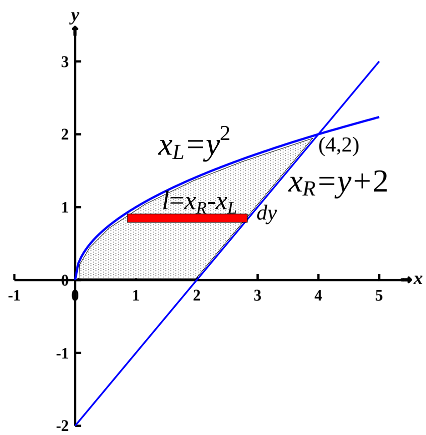
The area of the second typical rectangle

$$dA_2 = (y_T - y_B)dx = (\sqrt{x} - (x-2))dx = (\sqrt{x} - x + 2)dx$$

$$\therefore A_2 = \int dA_2 = \int_2^4 (\sqrt{x} - x + 2)dx = \left.\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right|_2^4$$

$$= [\frac{4^{3/2}}{3/2} - \frac{4^2}{2} + 2*4] - [\frac{2^{3/2}}{3/2} - \frac{2^2}{2} + 2*2] = 1.447715$$

$\therefore A = 1.885618 + 1.447715 = 3.333333$ square units



b. Using horizontal strip:

The area of the typical rectangle

$$dA = (x_R - x_L)dy = \{(y+2) - y^2\}dy$$

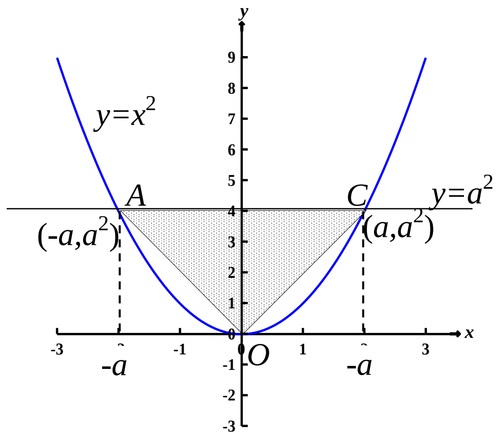
$$\therefore A = \int dA = \int_0^2 (y + 2 - y^2)dy = \left.\frac{y^2}{2} + 2y - \frac{y^3}{3}\right|_0^2$$

$= [\frac{2^2}{2} + 2*2 - \frac{2^3}{3}] - [0] = 3.33333$ square units

Homework:

1. Find the area between $y=x$ and $y=x^3$ from x=-1 to x=1.
2. Find the area of the "triangle" region bounded by the y-axis and the curves $y=\sin x$ and $y=\cos x$ in the first quadrant.

3. Find the area bounded on the right by $x+y=2$, and on the left by $y=x^2$ and below by x-axis.
4. The area of the region between the curve $y=x^2$ and the line $y=4$ is divided into equal partitions by the line $y=c$.
   a. Find c by integrating with respect to y. (This puts c into the limits of integration).
   b. Find c by integrating with respect to x. (This puts c into the integrand as well)
5. Figure below shows triangle AOC inscribed in the region cut from the parabola $y=x^2$ by the line $y=a^2$. Find the ratio of the area of the triangle to the area of parabolic region.



6. Find the area bounded by:
   a. The curve $y=4x-x^2$ and the lines $y=0$, $x=1$ and $x=3$.
   b. The curve $x=1+y^2$ and the line $x=10$.
   c. The curve $y=9-x^2$ and the line $y=x+3$.
   d. The curves $y=x^2-4$, $y=8-2x^2$.
   e. The curve $y=x^2-2$ and the line $y=2$
   f. The curve $y=x^2-2x$ and the line $y=x$.
   g. The curve $x=3y-y^2$ and the line $x+y=3$
   h. The curves $y=x^2$, $y=-x^2+4$
   i. The curves $y=\cos\frac{\pi x}{2}$, $y=1-x^2$ from $x=0$ to $x=1$
   j. The curve $y=\sin\frac{\pi x}{2}$ and the line $y=x$ from $x=-1$ to $x=1$

syllabus

Mathematics