Rules for Definite Integrals:

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b k.f(x)dx = k\int_a^b f(x)dx$ where k is constant

   when k=-1, then, $\int_a^b (-1)f(x)dx = -\int_a^b f(x)dx$

4. $\int_a^b [f(x) \mp g(x)]dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx$

5. If $f(x) \geq 0$ on $[a, b]$ then, $\int_a^b f(x)dx \geq 0$ on $[a, b]$

6. If $f(x) \geq g(x)$ on $[a, b]$ then, $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ on $[a, b]$

7. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ where $a \leq c \leq b$

Example 1: Evaluate the following integrals:

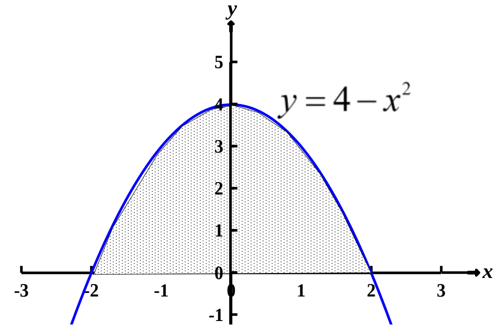
a. $\int_0^\pi \cos x dx = \sin x\Big|_0^\pi = \sin\pi - \sin 0 = 0 - 0 = 0$

b. $\int_0^2 x^2 dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{1}{3}(2^3 - 0^3) = \frac{1}{3}(8-0) = \frac{8}{3}$

c. $\int_{-2}^2 (4 - x^2)dx = \left.(4x - \frac{x^3}{3})\right|_{-2}^2 = [4(2) - \frac{(2)^3}{3}] - [4*(-2) - \frac{(-2)^3}{3}]$

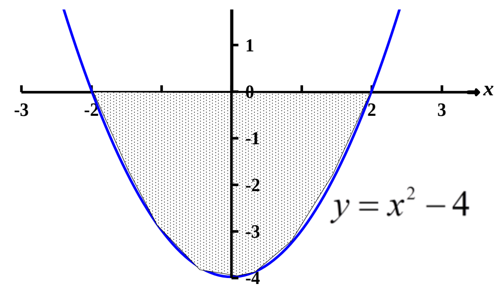
$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{32}{3}$$

Example 2: Find the area between the x-axis and the curve: a. $y = 4 - x^2$ b. $y = x^2 - 4$ for $-2 \le x \le 2$



a. since $y = 4 - x^2 \ge 0$ on [-2,2], the area between the curve and x-axis from -2 to 2:

$$Area = \int_{-2}^{2} (4 - x^2)dx = (4x - \frac{x^3}{3})\Big|_{-2}^{2} = \frac{32}{3} \text{ square units}$$



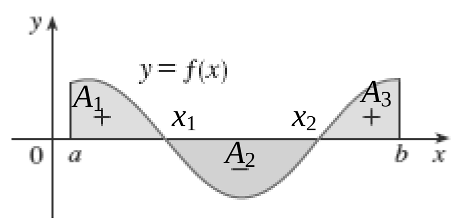
b. since $y = x^2 - 4 \le 0$ on [-2, 2], the area between the curve and x-axis from -2 to 2:

$$Area = -\int_{-2}^{2} (x^2 - 4)dx = \int_{-2}^{2} -(x^2 - 4)dx$$

$$= \int_{-2}^{2} (4 - x^2)dx = \frac{32}{3} \text{ square units.}$$

Steps for finding area when *f* has both positive and negative values

1. Find the points where f=0.
2. Use the zeroes of f to partition $[a, b]$ into sub intervals.
3. Integrate over such intervals.
4. Add the absolute values of the results.



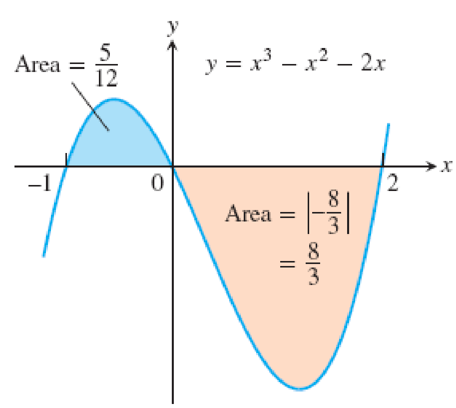
So $A = A_1 + A_2 + A_3 = \left|\int_a^{x_1} f(x)dx\right| + \left|\int_{x_1}^{x_2} f(x)dx\right| + \left|\int_{x_2}^{b} f(x)dx\right|$

Example: Find the area of the region between the x-axis and the graph of

$$f(x) = x^3 - x^2 - 2x, -1 \le x \le 2.$$

Sol.: 1. Find the zeroes of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x-2)(x+1)$$



Put $x(x-2)(x+1)=0 \quad \Rightarrow x=0$ or $x=2$ or $x=-1$

2. Use the zeroes of *f* to partition $[a, b]$ into subintervals.

P={-1, 0, 2}

3. Integrate over such intervals.

$$\int_{-1}^{0}(x^3-x^2-2x)dx=\left.\frac{x^4}{4}-\frac{x^3}{3}-\frac{2x^2}{2}\right|_{-1}^{0}$$

$$=\left(\frac{0^4}{4}-\frac{0^3}{3}-0^2\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^3}{3}-(-1)^2\right)=\frac{5}{12}$$

$$\int_{0}^{2}(x^3-x^2-2x)dx=\left.\frac{x^4}{4}-\frac{x^3}{3}-x^2\right|_{0}^{2}$$

$$=\left(\frac{2^4}{4}-\frac{2^3}{3}-2^2\right)-0=-\frac{8}{3}$$

4. Add the absolute values of the results.

$$A=|A_1|+|A_2|=\left|\frac{5}{12}\right|+\left|-\frac{8}{3}\right|=\frac{5}{12}+\frac{8}{3}=\frac{37}{12}$$ square units.

Homework:

1. Evaluate the following integrals.

a. $\int_{1}^{2}(2x+5)dx$ b. $\int_{0}^{\pi/3}2\sec^2 x dx$ c. $\int_{\pi/4}^{3\pi/4}\csc x\cot x dx$

d. $\int_{-1}^{1}(r+1)^2 dr$ e. $\int_0^{\pi}(1+\cos x)dx$ f. $\int_{0}^{\pi/4}\frac{1-\sqrt{u}}{\sqrt{u}}du$

2. Find the total area of the region between the curve and the *x*-axis.

a. $y=2-x$; $0\le x\le 3$ b. $y=3x^2-3$; $-2\le x\le 2$

c. $y=x^3-3x^2$; $0\le x\le 2$ d. $y=x^3-4x$; $-2\le x\le 2$

3. Find the derivatives of the following in two ways:

-by evaluating the integral and differentiating the results, and

- by applying the first fundamental theorem.

a. $\frac{d}{dx}\int_{0}^{x}\cos t\,dt$ b. $\frac{d}{dx}\int_{0}^{\sin x}3t^2\,dt$

4. Find dy/dx of the following:

a. $y=\int_{0}^{x}\sqrt{1+t^2}\,dt$ b. $y=\int_{0}^{x}\frac{1}{t}.dt$

c. $y=\int_{0}^{x}\sin t^2\,dt$ d. $y=\int_{0}^{2x}\cos t\,dt$

5. Suppose f and g are continuous functions and that:

$\int_{1}^{2}f(x)dx=-4$, $\int_{1}^{5}f(x)dx=6$ and $\int_{1}^{5}g(x)dx=8$ use the properties of definite integral to find.

a. $\int_{2}^{5}f(x)dx$ b. $\int_{1}^{5}\left(f(x)-g(x)\right)dx$ c. $\int_{1}^{2}3f(x)dx$

d. $\int_{2}^{2}g(x)dx$ e. $\int_{5}^{1}g(x)dx$ f. $\int_{1}^{5}\left(4f(x)-g(x)\right)dx$