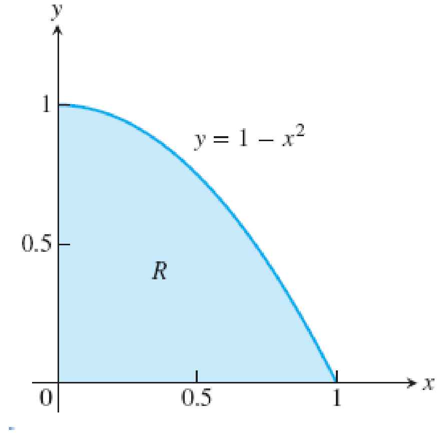
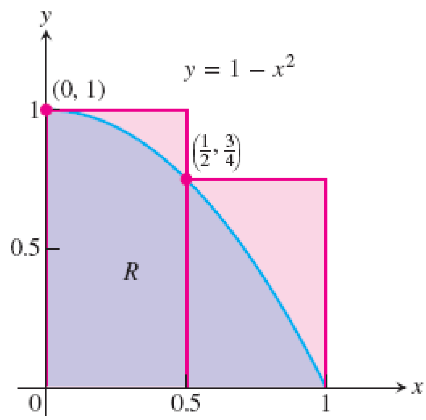
# Definite Integrals:

## Area under Curve:

The area of the region with a curved boundary can be approximated by summing the areas of a collection of rectangles. Using more rectangles can increase the accuracy of approximation.




***Example 1:*** What is the area of the shaded region $R$ that lies above the $x$-axis, below the graph of $y=1-x^2$, and between the vertical lines $x=0$ and $x=1$?




**Sol.:** 1. If we divide the region into two rectangles (two sub-intervals each sub interval length $\Delta x=\frac{b-a}{n}=\frac{1-0}{2}=\frac{1}{2}$), and we take the height of rectangles at the left-end of those sub-intervals.

$$\therefore A\approx\frac{1}{2}*1+\frac{1}{2}*\frac{3}{4}=\frac{7}{8}=0.875$$

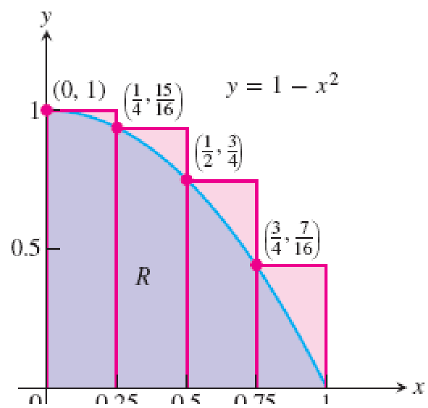
We get an upper estimated of the area $A$.

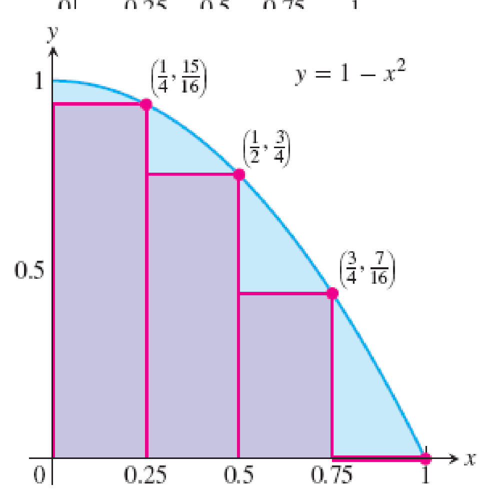
If we use four rectangles we get a better upper estimate.

$$\therefore \Delta x=\frac{b-a}{n}=\frac{1-0}{4}=\frac{1}{4}$$

and $$\therefore A\approx\frac{1}{4}(1+\frac{15}{16}+\frac{3}{4}+\frac{7}{16})=\frac{25}{32}=0.78125$$

which is still greater than the area $A$ since the rectangles all lies outside the region $R$.



2. If we take the height of rectangles at the right-end of those sub-intervals,

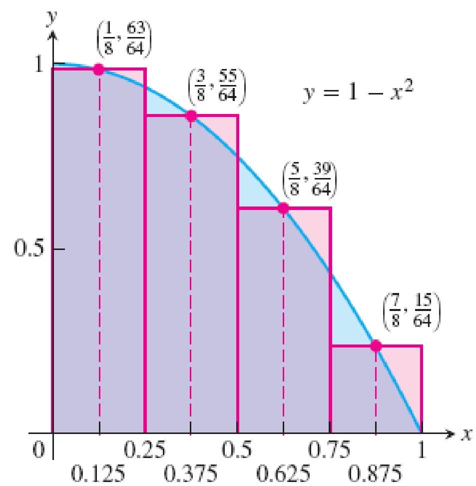
$$\therefore A \approx \frac{1}{4}(\frac{15}{16} + \frac{3}{4} + \frac{7}{16} + 0) = \frac{17}{32} = 0.53125$$

This estimate is smaller than the area $A$ since the rectangles all lies inside the region $R$.

So, the true value of $A$ lies somewhere between these lower and upper sums;

$$0.53125 < A < 0.78125$$

3. Another estimate can be obtained by using rectangles whose heights are the values of $f$ at the midpoint of their bases. This gives an estimate that is between the lower sum and upper sum, but is not clear whether it overestimates or underestimates the true area.



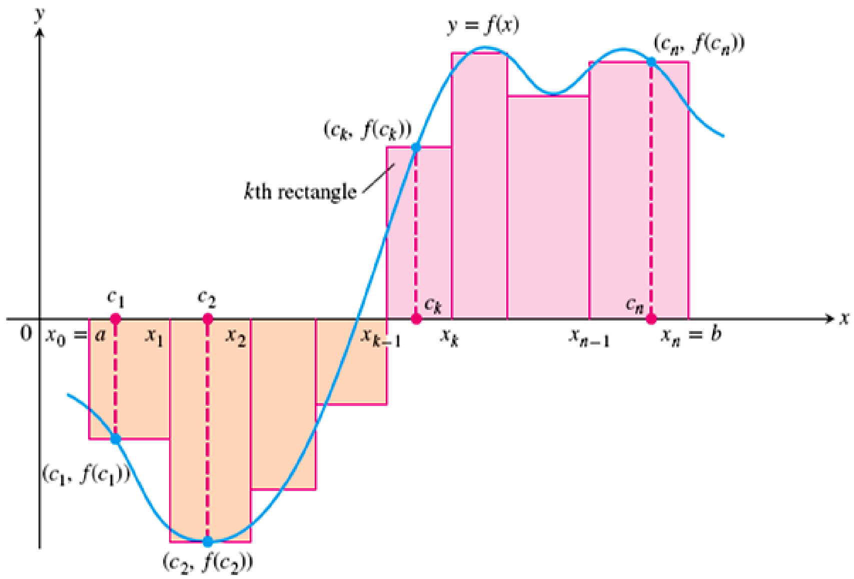

$$\therefore A \approx \frac{1}{4}(\frac{63}{64} + \frac{55}{64} + \frac{39}{64} + \frac{15}{64}) = \frac{1}{4} * \frac{172}{64} = 0.671875$$

If we increase the subdivisions we obtain:

| **Number of Sub-intervals** | **Lower sum** | **Midpoint sum** | **Upper sum** |
|---|---|---|---|
| **2** | 0.375 | 0.6875 | **0.875** |
| **4** | 0.53125 | 0.671875 | **0.78125** |
| **16** | 0.634765625 | 0.6669921875 | **0.697285625** |
| **50** | 0.6566 | 0.6667 | **0.6766** |
| **100** | 0.66165 | 0.666675 | **0.67165** |
| **1000** | **0.6661665** | **0.66666675** | **0.6671665** |

## Riemann Sums:

Suppose $y=f(x)$ is an arbitrary continuous function over closed interval $[a, b]$, $f(x)$ may have negative as well as positive values.




We subdivide the interval $[a, b]$ into subintervals not necessary of equal width (length). To do so, we choose $n$-1 points $\{x_1, x_2, x_3\ldots x_{n-1}\}$ between $a$ and $b$ and satisfying

$$a=x_o < x_1 < x_2 <\ldots< x_{n-1}< x_n =b$$

the set

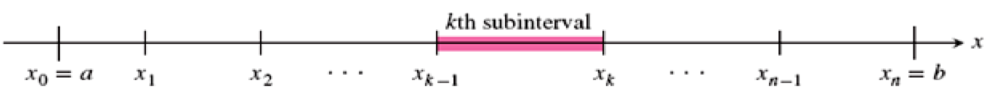
$$P=\{x_1, x_2, x_3\ldots x_{n-1}\}$$

is called a **partition** of $[a, b]$

The partition $P$ divides $[a, b]$ into $n$ closed sub intervals,

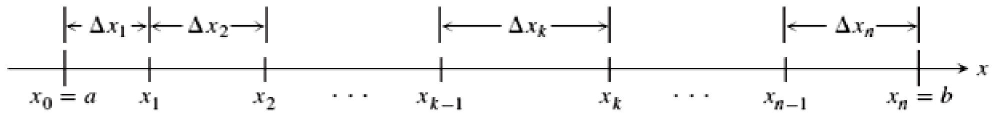
$$[x_o, x_1] , [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

The first of these subintervals is $[x_o, x_1]$, the second is $[x_1, x_2]$ and the $\boldsymbol{k^{th}}$ **subintervals** of $P$ is $[x_{k-1}, x_k]$, for $k$ an integer between 1 and $n$.



The width of the first subintervals $[x_o, x_1]$ is denoted $\Delta x_1$, the width of the second $[x_1, x_2]$ is $\Delta x_2$, and the width of the $k^{th}$ is $\Delta x_k = x_k - x_{k-1}$. If all $n$ subintervals have the equal width, then the common width $\Delta x$ is equal to

$$\Delta x = \frac{b-a}{n}$$




In each subinterval we select some point. The point chosen in the $k^{th}$ subinterval is called $c_k$. Then on each subinterval we stand a vertical rectangle that stretches from the $x$-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the $x$-axis, depending whether $f(c_k)$ is positive or negative, or on it $f(c_k)=0$.

On each subinterval we form the product $f(c_k).\,\Delta x_k$. This product is positive, negative or zero, depending on the sign of $f(c_k)$.

If $f(c_k) > 0$, the product $f(c_k).\,\Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width $\Delta x_k$.

If $f(c_k) < 0$, the product $f(c_k).\,\Delta x_k$ is a negative number, the negative of the area of a rectangle with height $f(c_k)$ and width $\Delta x_k$ that drops from the $x$-axis to the negative number of $f(c_k)$.

Finally we sum all of these products to get

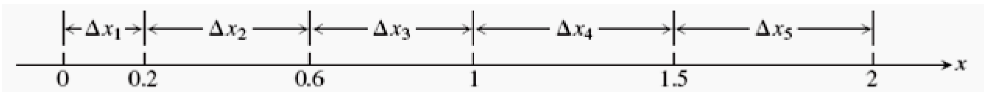
$$S_P = \sum_{k=1}^{n} f(c_k).\Delta x_k$$

The sum $S_P$ is called a **Riemann Sum** for $f$ on the interval $[a, b]$. There are many such sums, depending on the partition $P$ we choose, and the choice of the point $c_k$ in the subintervals.

We define the **norm** of a partition $P$, written $\|P\|$, to be the largest of all subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition $P$ have a small width.

**Example 1:** Partitioning a Closed Interval.

The set $P$ = {0. 0.2, 0.6, 1, 1.5, 2} is a partition of [0,2]. There are five subintervals $P$: [0, 0.2], [0.2, 0.6], [0.6, 1], [1, 1.5] and [1.5, 2]




The lengths of the subintervals are: $\Delta x_1 = 0.2 - 0 = 0.2$,

$\Delta x_2 = 0.6 - 0.2 = 0.4$

$\Delta x_3 = 0.4$

$\Delta x_4 = 0.5$

and $\Delta x_5 = 0.5$

The longest subinterval length is 0.5, so the norm of the partition $\|P\|=0.5$. In this example there are two subintervals of this length.

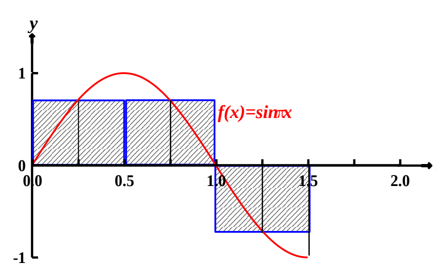
Any Riemann sum associated with a partition of a closed interval $[a,b]$ defines rectangles that approximate the region between the graph of a continuous function $f$ and the $x$-axis. Partitions with norm approaching to zero lead to collections of rectangles that approximate this region with increasing accuracy.

**Example 2:** Find the Riemann sum for $f(x) = \sin \pi x$ on the interval $[0, 3/2]$. Use $n$=3.

**Sol.:** $\Delta x = \frac{b-a}{n} = \frac{3/2 - 0}{3} = \frac{1}{2} \quad \Rightarrow \quad x_0 = a = 0,\ x_1 = \frac{1}{2},\ x_2 = 1$ and $x_3 = b = \frac{3}{2}$

1. Choice of $c_k$: the interval midpoint

$$c_1 = \frac{1/2 + 0}{2} = \frac{1}{4},\ c_2 = \frac{1 + 1/2}{2} = \frac{3}{4},\ c_3 = \frac{3/2 + 1}{2} = \frac{5}{4}$$



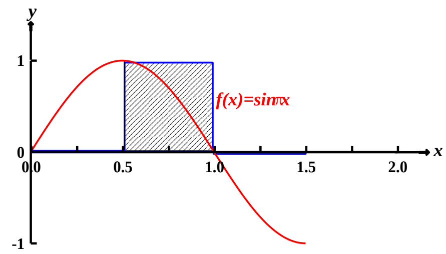

The Riemann sum:

$$\sum_{k=1}^{3} f(c_k)\Delta c_k = \sum_{k=1}^{3} \sin \pi c_k * \frac{1}{2} = \frac{1}{2}(\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4})$$

$$= \frac{1}{2}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{4}$$

2. Choice of $c_k$: left-hand of subinterval

$$c_1 = 0\,,\ c_2 = \frac{1}{2}\,,\ c_3 = 1$$



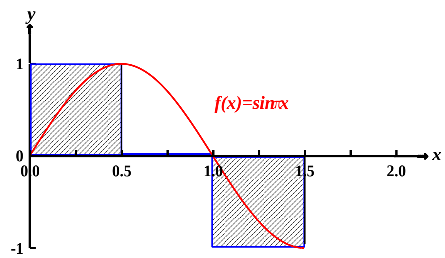

The Riemann sum:

$$\sum_{k=1}^{3} f(c_k)\Delta c_k = \sum_{k=1}^{3} \sin \pi c_k * \frac{1}{2} = \frac{1}{2}(\sin 0 + \sin\frac{\pi}{2} + \sin\pi)$$

$$= \frac{1}{2}(0+1+0) = \frac{1}{2}$$

3. Choice of $c_k$: right-hand of subinterval

$$c_1 = \frac{1}{2}\,,\ c_2 = 1\,,\ c_3 = \frac{3}{2}$$




The Riemann sum:

$$\sum_{k=1}^{3} f(c_k)\Delta c_k = \sum_{k=1}^{3} \sin \pi c_k * \frac{1}{2} = \frac{1}{2}(\sin\frac{\pi}{2} + \sin\pi + \sin\frac{3\pi}{2})$$

$$= \frac{1}{2}(1+0-1) = 0$$

## Area Is Strictly a Special Case

If an integerable function $y=f(x)$ is nonnegative throughout an interval $[a, b]$, each term $f(c_k)\Delta x_k$ is the area of a rectangle reaching from the $x$-axis up to the curve $y=f(x)$. The Riemann sum

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

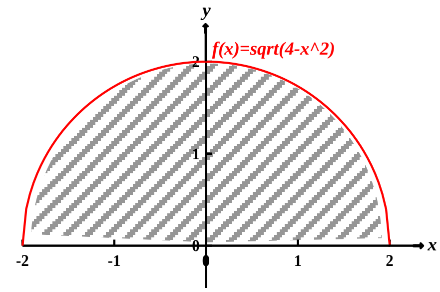
which is the sum of the areas of these rectangles, gives an estimate of the area of the region between the curve and the $x$-axis from $a$ to $b$. Since the rectangles give an increasing good approximation of the region as we use subdivisions with smaller and smaller subintervals, we call the limiting value

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_x = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_x = \int_a^b f(x)dx$$ the area under the curve.

**Note:** (remember that when $n\to\infty \Rightarrow \Delta x\to 0$)

***Definition:*** If $y=f(x)$ is nonnegative and integerable function over a closed interval $[a, b]$, then the integral of $f$ from $a$ to $b$ is the **area** of the region between the graph of $f$ and the $x$-axis from $a$ to $b$. We sometimes call this number the **area under the curve $y=f(x)$ from $a$ to $b$.**

***Example 3:*** Find the value of integral $\int_{-2}^{2}\sqrt{4-x^2}\,dx$ by regarding it as the area under the graph of an appropriately chosen function.




**Sol.:** We graph the integrand $f(x)=\sqrt{4-x^2}$ over the interval of integration $[-2, 2]$ and see that the graph is semicircle of radius $2$. The area between the semicircle and the $x$-axis is

$$Area=\frac{1}{2}\pi r^2=\frac{1}{2}\pi(2)^2=2\pi$$

Because the area is also the value of the integral of $f$ from $-2$ to$2$:

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx=2\pi$$

### The First Fundamental Theorem of Calculus

If the function $f$ is continuous on an interval $[a, b]$, and $F(x)$ is the antiderivative of $f$, then the function $F(x)=\int_{a}^{x} f(t)dt$ has a derivative at every point on $[a, b]$ and

$$\frac{dF}{dx}=\frac{d}{dx}\int_{a}^{x} f(t)dt=f(x)$$

***Examples:*** Find $dy/dx$ of the following:

syllabus

1. $y = \int_{-\pi}^{x} \cos t.dt$

**Sol.:** $\frac{d}{dx}\int_{-\pi}^{x} \cos t.dt = \cos x$

2. $y = \int_{0}^{x} \frac{1}{1+t^2}.dt$

**Sol.:** $\frac{d}{dx}\int_{0}^{x} \frac{1}{1+t^2}.dt = \frac{1}{1+x^2}$

3. $y = \int_{1}^{x^2} \cos t.dt$

**Sol.:** $\frac{d}{dx}\int_{1}^{x^2} \cos t.dt = \cos x^2 * \frac{d}{dx}(x^2) = \cos x^2 * 2x = 2x\cos x^2$

**The Second Fundamental Theorem of Calculus** (The Integral Evaluation Theorem)

If $f$ is continuous at every point of $[a, b]$ and $F$ is antiderivative of $f$ on $[a,b]$, then

$$\int_{a}^{b} f(x)dx = F(x)\Big|_{a}^{b} = F(b) - F(a)$$