INTEGRATION

Integration process is the reverse of differentiation process.

1. Indefinite Integrals:

If $\frac{d}{dx}F(x) = f(x)$ or $F`(x) = f(x)$ then $dF(x) = f(x).dx$

Note: If $f(x)$ is any function then $F(x)$ is called the antiderivative of $f(x)$.

$$\therefore \int dF(x) = \int f(x).dx$$

But since: $d(F(x)+C) = dF(x) = f(x).dx$

Hence: $\int f(x).dx = \int d(F(x)+C)$

$$\therefore \int f(x).dx = F(x)+C$$

Where C: is called constant of integration,

and $\int f(x).dx$: is indefinite integral of the function $f(x)$ with respect to x.

Rules for indefinite integrals:

1. $\int \frac{dF(x)}{dx}.dx = F(x)+C$
2. $\frac{d}{dx}\int f(x).dx = f(x)$
3. $\int x^n.dx = \frac{x^{n+1}}{n+1}+C$ where $n \neq -1$
4. $\int k.f(x).dx = k\int f(k).dx$ where k is constant (Does not work if k varies with x)
5. $\int [f(x) \mp g(x)].dx = \int f(x).dx \mp \int g(x).dx$
6. $\int \sin x.dx = -\cos x + C$ or $\int \sin u.du = -\cos u + C$ where u is function of x.
7. $\int \cos x.dx = \sin x + C$ or $\int \cos u.du = \sin u + C$
8. $\int \sec^2 x.dx = \tan x + C$ or $\int \sec^2 u.du = \tan u + C$

9. $\int \csc^2 x.dx = -\cot x + C$ or $\int \csc^2 u.du = -\cot u + C$

10. $\int \sec x \tan x.dx = \sec x + C$ or $\int \sec u \tan u.du = \sec u + C$

11. $\int \csc x \cot x.dx = -\csc x + C$ or $\int \csc u \cot u.du = -\csc u + C$

Examples: Evaluate the following integrals.

1. $\int x^5.dx = \frac{x^6}{6} + C$

2. $\int \sin 2x.dx$

Sol.: let $u=2x \Rightarrow du=2dx \Rightarrow dx=du/2$

$$\int \sin 2x.dx = \int \sin u.\frac{du}{2} = \frac{1}{2}(-\cos u) + C = \frac{-\cos 2x}{2} + C$$

3. $\int \cos\frac{x}{2}.dx$

Sol.: let $u = \frac{x}{2} \Rightarrow du = \frac{dx}{2} \Rightarrow dx = 2du$

$$\int \cos\frac{x}{2}.dx = \int \cos u.(2du) = 2\int \cos u.du = 2\sin u + C = 2\sin\frac{x}{2} + C$$

4. $\int (x+5)^5.dx$

Sol.: let $u = x+5 \Rightarrow du = dx$

$$\int u^5.du = \frac{u^6}{6} + C = \frac{(x+5)^6}{6} + C$$

5. $\int \sqrt{4x-1}.dx$

Sol.: let $u = 4x-1 \Rightarrow du = 4dx \Rightarrow dx = \frac{du}{4}$

$$\int u^{1/2}.\frac{du}{4} = \frac{1}{4}\frac{u^{3/2}}{3/2} + C = \frac{2}{12}u^{3/2} + C = \frac{1}{6}u^{3/2} + C = \frac{1}{6}(4x-1)^{3/2} + C$$

6. $\int \cos(7x+5).dx$

Sol.: let $u = 7x+5 \Rightarrow du = 7dx \Rightarrow dx = \frac{du}{7}$

$$\int \cos u.\frac{du}{7}=\frac{1}{7}\sin u+C=\frac{1}{7}\sin(7x+5)+C$$

7. $\int x^2 \sin x^3.dx$

Sol.: let $u=x^3 \Rightarrow du=3x^2dx \Rightarrow x^2dx=\frac{du}{3}$

$$\int \sin u.\frac{du}{3}=\frac{-\cos u}{3}+C=\frac{-\cos x^3}{3}+C$$

8. $\int \frac{1}{\cos^2 2x}.dx$

Sol.: $\int \frac{1}{\cos^2 2x}.dx=\int \sec^2 2x.dx$

let $u=2x \Rightarrow du=2dx \Rightarrow dx=\frac{du}{2}$

$$\int \sec^2 u.\frac{du}{2}=\frac{1}{2}\tan u+C=\frac{1}{2}\tan 2x+C$$

9. $\int (x^2+2x-3)^2(x+1).dx$

Sol.: let $u=x^2+2x-3 \Rightarrow du=(2x+2)dx=2(x+1)dx \Rightarrow (x+1)dx=\frac{du}{2}$

$$\int u^2.\frac{du}{2}=\frac{1}{2}*\frac{u^3}{3}+C=\frac{u^3}{6}+C=\frac{(x^2+2x-3)^3}{6}+C$$

10. $\int \sin^4 x\cos x.dx$

Sol.: let $u=\sin x \Rightarrow du=\cos x.dx$

$$\int u^4.du=\frac{u^5}{5}+C=\frac{\sin^5 x}{5}+C$$

11. $\int \frac{2zdz}{\sqrt[3]{z^2+1}}$

Sol.: let $u=z^2+1 \Rightarrow du=2zdz$

$$\int \frac{du}{u^{1/3}}=\int u^{-1/3}.du=\frac{u^{2/3}}{2/3}+C=\frac{3}{2}u^{2/3}+C=\frac{3}{2}(z^2+1)^{2/3}+C$$

Another solution:

let $u=\sqrt[3]{z^2+1} \Rightarrow u^3=z^2+1 \Rightarrow 3u^2du=2zdz$

$$\int \frac{2zdz}{\sqrt[3]{z^2+1}}=\int \frac{3u^2du}{u}=\int 3u.du=\frac{3u^2}{2}+C=\frac{3}{2}(z^2+1)^{2/3}+C$$

12. $\int \tan x \sec^2 x.dx$

Sol.: let $u=\tan x \Rightarrow du=\sec^2 x.dx$

$$\int u.du=\frac{u^2}{2}+C=\frac{\tan^2 x}{2}+C$$

Another solution:

let $u=\sec x \Rightarrow du=\sec x \tan x.dx$

$$\int \tan x \sec^2 x.dx=\int \sec x(\sec x \tan xdx)=\int u.du=\frac{u^2}{2}+C=\frac{\sec^2 x}{2}+C$$

13. $\int 3x^2\sqrt{x^3+1}.dx$

Sol.: let $u=x^3+1 \Rightarrow du=3x^2dx$

$$\int \sqrt{u}.du=\frac{u^{3/2}}{3/2}+C=\frac{2}{3}(x^3+1)^{3/2}+C$$

Another solution:

let $u^2=x^3+1 \Rightarrow 2udu=3x^2dx$

$$\int u*2udu=\int 2u^2.du=2\frac{u^3}{3}+C=\frac{2}{3}(x^3+1)^{3/2}+C$$

14. $\int \frac{x^2+2x}{x^2+2x+1}dx$

Sol.: When the degree of numerator is equal or greater than the degree of denominator, use long division to simplify the problem.

$$\begin{array}{r|l} & 1 \\ x^2+2x+1 & x^2+2x \\ & \mp x^2 \mp 2x \mp 1 \\ \hline & 0+0 \quad -1 \end{array}$$

$$\int \frac{x^2+2x}{x^2+2x+1}dx=\int\left(1-\frac{1}{x^2+2x+1}\right)dx=\int\left(1-\frac{1}{(x+1)^2}\right)dx$$

$$=\int\left(1-(x+1)^{-2}\right)dx=x-\frac{(x+1)^{-1}}{-1}+C=x+\frac{1}{x+1}+C=\frac{x^2+x+1}{x+1}+C$$

15. $\int \frac{(1+x)^2}{\sqrt{x}}.dx$

Sol.: $\int \frac{(1+x)^2}{\sqrt{x}}.dx = \int \frac{1+2x+x^2}{\sqrt{x}}\,dx = \int \left(\frac{1}{\sqrt{x}} + \frac{2x}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right)dx = \int \left(x^{-1/2} + 2x^{1/2} + x^{3/2}\right)dx$

$$= \frac{x^{1/2}}{1/2} + \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} + C = 2x^{1/2} + \frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} + C$$

The Integrals of $\sin^2 x$ and $\cos^2 x$:

use the following identities: $\boxed{\sin^2 x = \frac{1-\cos 2x}{2}}$ $\boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$

16. $\int \sin^2 x.dx = \int \frac{1-\cos 2x}{2}\,dx = \int \frac{1}{2}\,dx - \int \frac{\cos 2x}{2}\,dx$

$$= \frac{x}{2} - \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

17. $\int \cos^2 2x.dx = \int \frac{1+\cos 4x}{2}\,dx = \int \frac{1}{2}\,dx + \int \frac{\cos 4x}{2}\,dx$

$$= \frac{x}{2} + \frac{1}{2}\left(\frac{\sin 4x}{4}\right) + C = \frac{x}{2} - \frac{\sin 4x}{8} + C$$

Solving Initial Value Problems with Indefinite Integrals

The problem of finding a function y of x when we know its derivative $\frac{dy}{dx} = f(x)$ and its value y_o at a particular point x_o is called *initial value problem.*

So, if $\frac{dy}{dx} = f(x) \Rightarrow dy = f(x).dx$

and $\int dy = \int f(x).dx$

$\therefore y = F(x) + C$ where $F(x)$ is the antiderivative of $f(x)$

In this case $y_o = F(x_o) + C \Rightarrow C = y_o - F(x_o)$ and $y = F(x) + \left(y_o - F(x_o)\right)$

Note: An equation like $\frac{dy}{dx} = f(x)$, that has a derivative in it, is called Differential Equation (*D.E.*).

A more complicated differential equation might involve *y* as well as *x*:

$\frac{dy}{dx} = 2xy^2$ first order *D.E.*

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 5y^3 = 3$ second order *D.E.*

$\frac{d^3y}{dx^3} + \frac{dy}{dx} + y^5 = x^2$ third order *D.E.*

Example 1: The velocity $v(t)$ of a body falling from *rest* in a vacuum near the surface of the earth satisfies

Differential equation: $\frac{dv}{dt} = 9.8$ (the acceleration is 9.8 m/sec^2)

Initial condition: v=0 when t=0 (the velocity is zero at start)

Find *v* as a function of ***t***.

Sol.: We find the general *D.E.* by integrating both sides of it with respect to ***t***:

$$a = \frac{dv}{dt} = 9.8 \quad (D.E.)$$

$$\int \frac{dv}{dt} dt = \int 9.8 \; dt \qquad \text{(integral equation)}$$

$$v + C_1 = 9.8t + C_2$$

$$v = 9.8t + C_2 - C_1$$

$$v = 9.8t + C \quad (\text{where } C = C_2 - C_1) \text{ (general solution)}$$

To find C use the boundary conditions: v=0 when t=0.

$$0 = 9.8*0 + C \quad \Rightarrow \quad C = 0$$

$$\therefore v = 9.8t + 0 \quad \Rightarrow \quad v = 9.8t$$

Example 2: Solve the following initial value problem for *y* as a function of *x*:

Differential equation: $\frac{d^2y}{dx^2}=6x-2$

Initial conditions (or boundary conditions): $\frac{dy}{dx}=0$ and $y=10$ when $x=1$

Sol.: *D. E.* $\frac{d^2y}{dx^2}=6x-2$

Integrate both sides:

$$\int\frac{d^2y}{dx^2}dx=\int(6x-2)dx \quad \rightarrow \quad \frac{dy}{dx}=\frac{6x^2}{2}-2x+C_1=3x^2-2x+C_1$$

We apply the first initial condition to find C_1 [$\frac{dy}{dx}=0$ when $x=1$]

$$0=3(1)^2-2(1)+C_1 \quad \Rightarrow \quad C_1=-1$$

This completes the formula for $\frac{dy}{dx}$:

$\frac{dy}{dx}=3x^2-2x-1$ also integrate both sides:

$$\int\frac{dy}{dx}dx=\int\left(3x^2-2x-1\right)dx \quad \rightarrow \quad y=x^3-x^2-x+C_2$$

We apply the second initial condition to find C_2 [$y=10$ when $x=1$]

$$10=(1)^3-(1)^2-(1)+C_2 \quad \Rightarrow \quad C_2=11$$

This completes the formula for y as a function of x.

$$y=x^3-x^2-x+11$$

Example 3: Evaluate the general solution of the following differential equations:

1. $\frac{dy}{dx}=x^2\sqrt{y}$

Sol.: Separate the variables:

$\frac{1}{\sqrt{y}}\frac{dy}{dx}=x^2$ Integrate both sides.

$$\int\frac{1}{\sqrt{y}}\frac{dy}{dx}dx=\int x^2dx$$

$$\frac{\sqrt{y}}{1/2}=\frac{x^3}{3}+C_1 \quad\Rightarrow\quad 2\sqrt{y}=\frac{x^3}{3}+C_1 \quad\Rightarrow\quad \sqrt{y}=\frac{x^3}{6}+\frac{C_1}{2}$$

$$\therefore y=\left[\frac{x^3}{6}+C\right]^2 \qquad \text{where} \qquad C=\frac{C_1}{2}$$

2. $\dfrac{dy}{dx}=\dfrac{\sqrt{x+1}}{\sqrt{y-1}}$

Sol.: $\sqrt{y-1}.dy=\sqrt{x+1}.dx$

$$\int\sqrt{y-1}.dy=\int\sqrt{x+1}.dx$$

$$\frac{(y-1)^{3/2}}{3/2}=\frac{(x+1)^{3/2}}{3/2}+C_1$$

$$(y-1)^{3/2}=(x+1)^{3/2}+\frac{3}{2}C_1 \quad\Rightarrow\quad y=\left[(x+1)^{3/2}+C\right]^{2/3}+1 \quad \text{where} \quad C=\frac{3}{2}C_1$$

3. $\dfrac{dy}{dx}=\sqrt{1+x+y+xy}$

Sol.: $\dfrac{dy}{dx}=\sqrt{1+x+y(1+x)} \quad\Rightarrow\quad \dfrac{dy}{dx}=\sqrt{(1+x)(1+y)} \quad\Rightarrow\quad \dfrac{dy}{dx}=\sqrt{1+x}.\sqrt{1+y}$

$$\frac{dy}{\sqrt{1+y}}=\sqrt{1+x}.dx \quad\Rightarrow\quad \int\frac{dy}{\sqrt{1+y}}=\int\sqrt{1+x}.dx \quad\Rightarrow\quad 2\sqrt{1+y}=\frac{2}{3}(1+x)^{3/2}+C_1$$

$$1+y=\left[\frac{1}{3}(1+x)^{3/2}+\frac{C_1}{2}\right]^2 \quad\Rightarrow\quad y=\left[\frac{1}{3}(1+x)^{3/2}+C\right]^2-1 \quad \text{where } C=\frac{C_1}{2}$$

Homework:

I. Evaluate the following integrals:

1. $\int\left(\dfrac{x^2}{2}+\dfrac{x^3}{3}\right).dx$ 2. $\int\dfrac{9r^2dr}{\sqrt{1-r^3}}$ 3. $\int\dfrac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

4. $\int\dfrac{x.dx}{(1+x^2)^2}$ 5. $\int(1+y)^{1/2}dy$ 6. $\int\sec^2(x+2)dx$

7. $\int \sqrt{\tan x}\sec^2 x.dx$ 8. $\int \sec^2\left(\frac{x}{4}\right)dx$ 9. $\int \frac{8\sin t}{\sqrt{5-4\cos t}}dt$

10. $\int \frac{\cos x}{\sqrt{2+\sin x}}dx$ 11. $\int 3\cos^2 x\sin x.dx$ 12. $\int (1-\sin 2t)^{3/2}\cos 2t.dt$

13. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ 14. $\int \tan^2 x\sec^2 x.dx$ 15. $\int \csc^2 2\theta\cot 2\theta.d\theta$

16. $\int \frac{18\tan^2 x\sec^2}{(2+\tan^3 x)^2}dx$ 17. $\int \sqrt{1+\sin^2(x-1)}.\sin(x-1)\cos(x-1)dx$

II. Solve the following initial value problems:

	Differential equation	Initial conditions
1.	$\frac{dy}{dx}=3\sqrt{x}$	y=4 when x=9
2.	$\frac{dy}{dx}=-\pi\sin\pi x$	y=0 when x=0
3.	$\frac{dy}{dx}=\frac{1}{2}\sec x\tan x$	y=1 when x=0
4.	$\frac{d^2y}{dx^2}=\frac{2}{x^3}$	$\frac{dy}{dx}=1$ and y=1 when x=1
5.	$\frac{d^3y}{dx^3}=6$	$\frac{d^2y}{d^2x}=-8$, $\frac{dy}{dx}=0$ and y=5 when x=0

III. A particle moves along a coordinate line with acceleration ($a=\frac{d^2s}{dt^2}=15\sqrt{t}-3/\sqrt{t}$), subject to conditions that $\frac{ds}{dt}=4$ and s=0 when t=1.

Find (a) the velocity ($v=\frac{ds}{dt}$) in terms of t.

(b) the position (s) in terms of t.

IV. The standard equation for free fall near the surface of every planet

$$s(t)=\frac{1}{2}gt^2+v_o t+s_o$$

Where $s(t)$ is the body's position on the line of fall, g is the planet (constant) acceleration of gravity, v_o is the body's initial velocity and s_o is the body's initial position.

Derive this equation by solving the following initial value problem:

Differential equation: $\frac{d^2s}{dt^2} = g$;

Initial conditions: $\frac{ds}{dt} = v_o$ and $s = s_0$ when $t=0$.

V. Show that:

1. $\int (\cos x - \sin x)^2 dx = x + \frac{\cos 2x}{2} + C$
2. $\int \frac{dx}{1+\cos 3x} = \frac{1-\cos 3x}{3\sin 3x} + C$
3. $\int \tan^5 x.\sec^2 x.dx = \frac{\tan^6 x}{6} + C$
4. $\int \frac{dt}{1-\sin\frac{t}{2}} = 2(\tan\frac{t}{2} + \sec\frac{t}{2}) + C$

VI. Evaluate the indefinite integrals:

1. $\int \sec 2x.\tan 2x.dx$
2. $\int \sec^2(x^2+2).2x.dx$
3. $\int \sqrt[3]{\tan x}.\sec^2 x.dx$
4. $\int \sin^4 x.\cos x.dx$