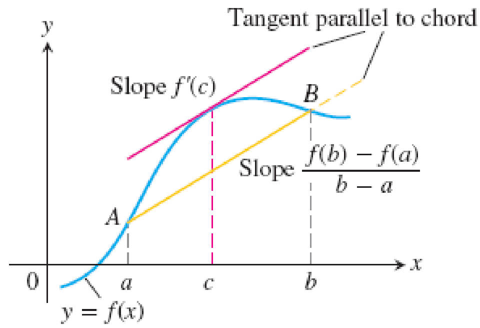
. The Mean Value Theorem (*M. V. T.*):

Suppose $y=f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), Then there is at least one point c in (a,b) at which

$$\frac{f(b)-f(a)}{b-a}=f`(c)$$



Examples: Does the *M. V. T.* be applicable on the following functions. If so find the value or values of c.

1. $f(x)=x-2\sin x;\qquad 0\le x\le 2\pi$

Sol.: 1. $f(x)=x-2\sin x$ is continuous on $[0, 2\pi]$.

2. $f`(x)=1-2\cos x$ is differentiable on $(0, 2\pi)$.

$\therefore$ The *M. V. T.* is applicable on $[0, 2\pi]$.

To find c:

$$f`(c)=\frac{f(b)-f(a)}{b-a}$$

where $f(b)=f(2\pi)=2\pi-2\sin 2\pi=2\pi-0=2\pi$

$f(a)=f(0)=0-2\sin 0=0-0=0$

and $f`(c)=1-2\cos c$, thus:

$$1-2\cos c=\frac{2\pi-0}{2\pi-0}\quad\Rightarrow\quad 1-2\cos c=1\quad\Rightarrow\quad 2\cos c=0\quad\Rightarrow\quad \cos c=0$$

$$\therefore c=\mp\frac{n\pi}{2};\qquad n=1,3,5..$$

$$\therefore c_1=\frac{\pi}{2}\quad\text{and}\quad c_2=\frac{3\pi}{2}$$ on the interval $[0, 2\pi]$.

2. $f(x)=x^{2/3};\qquad [-8, 8]$

Sol.: 1. $f(x)=x^{2/3}=\sqrt[3]{x^2}$ is continuous on $[-8, 8]$.

2. $f`(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3\sqrt[3]{x}}$ is not differentiable $x=0\in(-8,8)$.

$\therefore$ The *M. V. T.* is not applicable on [-8, 8].

3. $f(x)=x^{2/3}$; [0, 8]

Sol.: 1. $f(x)=x^{2/3}=\sqrt[3]{x^2}$ is continuous on [0, 8].

2. $f`(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3\sqrt[3]{x}}$ is not differentiable $x=0\notin(0,8)$

So it is differentiable on (0,8)

$\therefore$ The *M. V. T.* is applicable on [0, 8].

To find *c*:

$$f`(c)=\frac{f(b)-f(a)}{b-a}$$

where $f(a)=f(0)=0^{2/3}=0$

$f(b)=f(8)=8^{2/3}=4$

and $f`(c)=\frac{2}{3\sqrt[3]{c}}$, thus:

$$\frac{2}{3\sqrt[3]{c}}=\frac{4-0}{8-0}\Rightarrow\frac{2}{3\sqrt[3]{c}}=\frac{1}{2}\Rightarrow\sqrt[3]{c}=\frac{4}{3}\Rightarrow c=\left(\frac{4}{3}\right)^3=\frac{64}{27}=2.3704$$

Note: If $f`(x)$ is continuous on $[a,b]$, the Max.-Min. Theorem for continuous functions tells us that $f`$ has absolute maximum value ($_{max.}f`$) and absolute minimum value ($_{min.}f`$) on the interval, the equation:

$$f`(c)=\frac{f(b)-f(a)}{b-a}$$

gives us the inequality:

$$_{min.}f`\leq\frac{f(b)-f(a)}{b-a}\leq_{max.}f`$$

Example: Estimate $f(1)$ if $f`(x)=\frac{1}{5-x^2}$ and $f(0)=2$.

Sol.: $a=0 \Rightarrow f(a)=f(0)=2$

$b=1 \Rightarrow f(b)=f(1)=?$

$$_{\min.} f` \le \frac{f(b)-f(a)}{b-a} \le _{\max.} f`$$

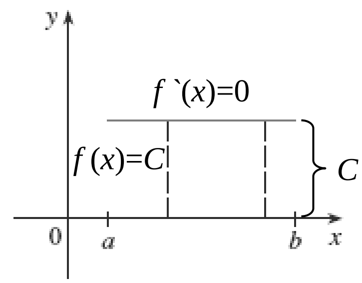
$$\frac{1}{5-0^2} \le \frac{f(1)-2}{1-0} \le \frac{1}{5-1^2}$$

$$\frac{1}{5} \le f(1)-2 \le \frac{1}{4}$$

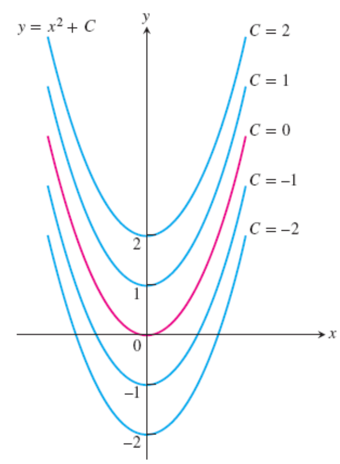
$$0.2+2 \le f(1) \le 0.25+2$$

$$2.2 \le f(1) \le 2.25$$

***Corollary^2*:** If $f`(x)=0$ for all x in an interval (a, b), then $f(x)=C$, for all $x\in (a, b)$, where C is a constant.



***Corollary^3*:** If $f`(x)=g`(x)$ for all in an interval (a, b), then f-g is constant on (a, b); that is $f(x) = g(x)+C$, where C is a constant.



Homework:

1. Show the following equations have exactly one solution in the given interval:

   $x^2 + 3x + 1 = 0$ $\qquad -2 \le x \le -1$

2. Find the value or values of c that satisfy the M.V.T. for the following functions and intervals:

   a. $f(x)=x^2+2x-1$ $\qquad 0 \le x \le 1$ $\qquad$ b. $f(x)=\sqrt{x-1}$ $\qquad 1 \le x \le 3$

c. $f(x) = x^{2/3}$ $\quad 0 \le x \le 1$ d. $f(x) = x + \frac{1}{x}$ $\quad \frac{1}{2} \le x \le 2$

3. By applying Mean Value Theorem, Show that for any number a and b

$$|\sin b - \sin a| \le |b - a|$$

4. By applying the inequality

$$_{\min} f \le \frac{f(b) - f(a)}{b - a} \le _{\max} f$$

Estimate $f(0.1)$ when

a. $f`(x) = \frac{1}{1 + x^4 \cos x}$ for $0 \le x \le 0.1$ and that $f(0)=1$.

b. $f`(x) = \frac{1}{1 - x^4}$ for $0 \le x \le 0.1$ and that $f(0)=2$.

5. Suppose that $f(0)=3$ and that $f`(x) = 0$ for all x. Use the Mean Value Theorem to show that $f(x)$ must be 3 for all x.

6. Suppose that $f`(x) = 2$ and that $f`(0) = 5$. Use the Mean Value Theorem to show that $f(x)=2x + 5$ at every value of x.