c. Horizontal and Vertical Asymptotes:

DEFINITION:

- A line $y=b$ is a horizontal asymptote of the graph of the function $y=f(x)$ if:

$$\text{either } \lim_{x\to+\infty} f(x)=b \quad \text{or} \quad \lim_{x\to-\infty} f(x)=b$$

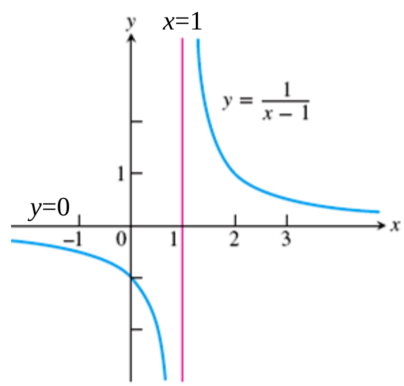
- A line $x=a$ is a vertical asymptote of the graph of the function $y=f(x)$ if:

$$\text{either } \lim_{x\to a^-} f(x)=\mp\infty \quad \text{or} \quad \lim_{x\to a^+} f(x)=\mp\infty$$

Examples: Find the asymptotes of the following curves:

1. $y=\dfrac{1}{x-1}$

Sol.: (a) horizontal asymptotes:



$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{1}{x-1}=\lim_{x\to\infty}\frac{1/x}{x/x-1/x}=\lim_{x\to\infty}\frac{1/x}{1-1/x}=\frac{0}{1-0}=0$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{1}{x-1}=\lim_{x\to-\infty}\frac{1/x}{x/x-1/x}=\lim_{x\to-\infty}\frac{1/x}{1-1/x}=\frac{0}{1-0}=0$$

$\therefore y=0$ (x-axis) is horizontal asymptote.

(b) vertical asymptotes:

To find a, put the denominator equal zero,

$$x-1=0 \quad \Rightarrow \quad x=1$$

$$\therefore a=1$$

$$\lim_{x\to a^-} f(x)=\lim_{x\to 1^-}\frac{1}{x-1}=\frac{1}{1^- -1}=-\infty$$

$$\lim_{x\to a^+} f(x)=\lim_{x\to 1^+}\frac{1}{x-1}=\frac{1}{1^+ -1}=+\infty$$

$\therefore x=1$ is vertical asymptote.

$$\begin{array}{r|l} & 1 \\ x+2 & x+3 \\ & \mp x \mp 2 \\ \hline & 1 \end{array}$$

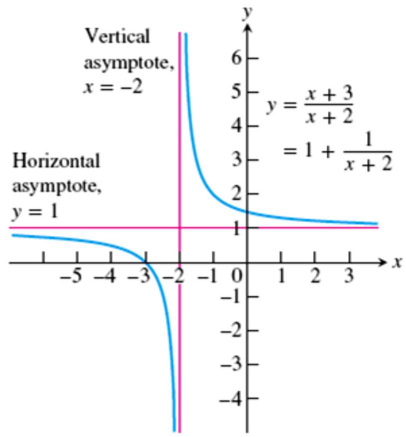
2. $y=\dfrac{x+3}{x+2}=1+\dfrac{1}{x+2}$

Sol.: (a) horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x+3}{x+2} = \lim_{x\to\infty}\frac{x/x+3/x}{x/x+2/x} = \lim_{x\to\infty}\frac{1+3/x}{1+2/x} = \frac{1+0}{1+0} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{x+3}{x+2} = \lim_{x\to-\infty}\frac{x/x+3/x}{x/x+2/x} = \lim_{x\to-\infty}\frac{1+3/x}{1+2/x} = \frac{1+0}{1+0} = 1$$

$\therefore y=1$ is horizontal asymptote.



(b) vertical asymptotes:

To find a, put the denominator equal zero,

$$x+2=0 \quad\Rightarrow\quad x=-2$$

$$\therefore a=-2$$

$$\lim_{x\to a^-} f(x) = \lim_{x\to -2^-}\frac{x+3}{x+2} = \frac{-2+3}{-2^-+2} = -\infty$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to -2^+}\frac{x+3}{x+2} = \frac{-2+3}{-2^++2} = +\infty$$

$\therefore x=-2$ is vertical asymptote.

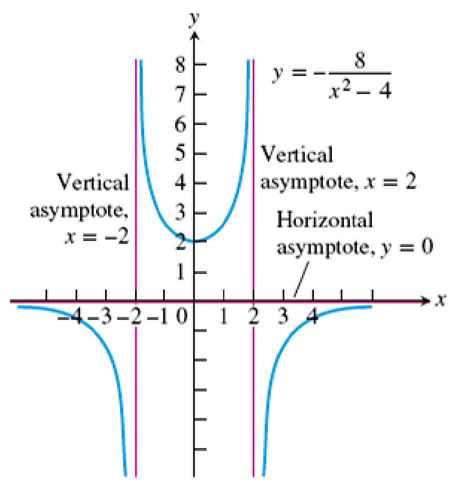
3. $y=\dfrac{-8}{x^2-4}$

Sol.: (a) horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{-8}{x^2-4} = \lim_{x\to\infty}\frac{-8/x^2}{x^2/x^2-4/x^2} = \lim_{x\to\infty}\frac{-8/x^2}{1-4/x^2} = \frac{0}{1-0} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{-8}{x^2-4} = \lim_{x\to-\infty}\frac{-8/x^2}{x^2/x^2-4/x^2} = \lim_{x\to-\infty}\frac{-8/x^2}{1-4/x^2} = \frac{0}{1-0} = 0$$

$\therefore y=0$ is horizontal asymptote.



(b) vertical asymptotes:

To find a, put the denominator equal zero,

$$x^2-4=0 \quad\Rightarrow\quad x=\mp 2$$

$$\therefore a=\mp 2$$

When $a=-2$ $\displaystyle\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}\frac{-8}{x^2-4} = \frac{-8}{(-2^-)^2-4} = -\infty$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{-8}{x^2-4} = \frac{-8}{(-2^+)^2-4} = +\infty$$

When $a=2$ $\quad \displaystyle\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{-8}{x^2-4} = \frac{-8}{(2^-)^2-4} = +\infty$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{-8}{x^2-4} = \frac{-8}{(2^+)^2-4} = -\infty$$

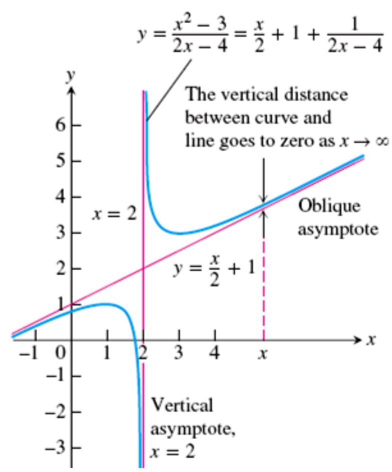
$\therefore x=-2$ and $x=2$ are vertical asymptotes.

d. Oblique Asymptotes:

If the degree of the numerator of a rational function is one greater than the degree of denominator, the graph has an oblique asymptote, that is, an asymptote that is neither vertical nor horizontal.

Example: Find the asymptotes of the curve:

$$y = \frac{x^2-3}{2x-4}$$



Sol.: To find asymptotes, oblique and otherwise, we divide $(2x-4)$ into (x^2-3):

$$\therefore y = \frac{x^2-3}{2x-4} = \underbrace{\frac{x}{2}+1}_{\text{linear}} + \underbrace{\frac{1}{2x-4}}_{\text{reminder}}$$

$$\begin{array}{r|l} & \frac{x}{2}+1 \\ \hline 2x-4 & x^2-3 \\ & \mp x^2 \pm 2x \\ & 2x-3 \\ & \mp 2x \pm 4 \\ \hline & 1 \end{array}$$

(a) Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-3}{2x-4} = \lim_{x\to\infty} \frac{x^2/x-3/x}{2x/x-4/x} = \frac{x-3/x}{2-4/x} = \frac{\infty-0}{+2-0} = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-3}{2x-4} = \lim_{x\to-\infty} \frac{x^2/x-3/x}{2x/x-4/x} = \frac{x-3/x}{2-4/x} = \frac{-\infty-0}{+2-0} = -\infty$$

So there is no horizontal asymptote.

(b) Vertical asymptotes:

To find *a*, put the denominator equal zero,

$$2x-4=0 \quad \Rightarrow \quad x=2$$

$$\therefore a=2$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-3}{2x-4} = \frac{2^2-3}{2*(2^-)-4} = -\infty$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2-3}{2x-4} = \frac{2^2-3}{2*(2^+)-4} = \infty$$

$\therefore x=2$ is vertical asymptote.

(c) Oblique asymptote:

As $x\to\pm\infty$, the reminder approaches zero and (the linear part is dominant) thus

$$f(x) \cong \frac{x}{2}+1.$$

So, the line $y=\frac{x}{2}+1$ is an oblique asymptote of the curve.

And as $x\to 2$, the reminder become large (the reminder part is dominant) thus

$$f(x) \cong \frac{1}{2x-4}$$

So, the line $x=2$ is a vertical asymptote of the curve.

Note: To find the oblique asymptote, do the following:

1. By long division, divide the equation of curve into two parts (linear part and reminder part)
2. Put *y* equal the linear part, so the resulted equation represent equation of inclined line and this line is the oblique asymptote of the curve.

Strategy for Graphing *y=f(x)*:

1. Identify the domain of *f*.
2. Identify any symmetry the curve may have.
3. Find *y*\` then find the critical points of *f*, and identify where the curve is increasing and where it is decreasing.

4. Find $y``$ then find the points of inflection, if any occur, and determine the concavity of the curve.
5. Identify any asymptotes.
6. Plot key points, such as intercepts and the points found in steps 3 and 4, and sketch the curve.

Example: Sketch the graph of $y = f(x) = \dfrac{x^2+1}{x}$.

Sol.: 1. Domain and Range of f.

-Domain: $x \neq 0 \;\Rightarrow\; D_f = (-\infty,\infty)\backslash\{0\}$

-Range: (put the function as $x=f(y)$).

$$y = \frac{x^2+1}{x} \;\Rightarrow\; xy = x^2+1$$

$$\Rightarrow\; x^2 - xy + 1 = 0$$

$A = 1,\; B = -y$ and $C = 1$

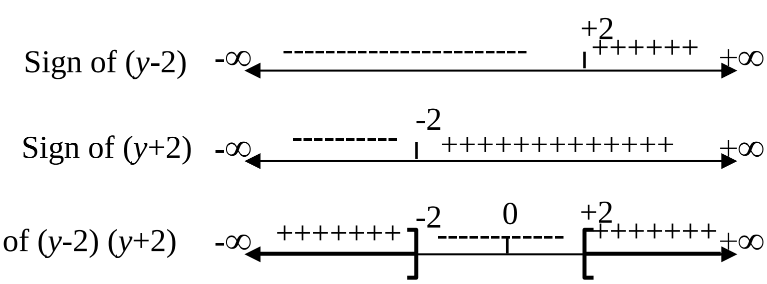
$$x = \frac{-B \mp \sqrt{B^2-4AC}}{2A}$$

$$= \frac{-(-y) \mp \sqrt{(-y)^2-4(1)(1)}}{2(1)}$$

$$= \frac{y \mp \sqrt{y^2-4}}{2}$$

$$\therefore y^2 - 4 \geq 0 \;\Rightarrow\; (y-2)(y+2) \geq 0$$

$$\therefore R_f = (-\infty,-2] \cup [2,\infty)$$



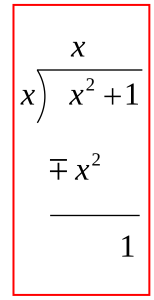
1. Symmetry:

$$f(-x) = \frac{(-x)^2+1}{(-x)} = -\frac{x^2+1}{x} \neq f(x)$$

$$-f(x) = -\left(\frac{x^2+1}{x}\right) = f(-x)$$

So the curve has symmetry about the origin.

2. First derivative test:



$$y = \frac{x^2+1}{x} = x + \frac{1}{x}$$

$$y` = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

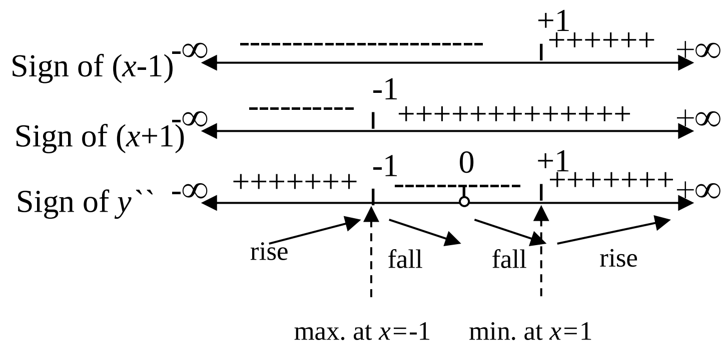
Put $y`=0 \quad \Rightarrow \quad \frac{x^2-1}{x^2} = 0 \quad \Rightarrow$

$$\frac{(x-1)(x+1)}{x^2} = 0$$

$\therefore$ either $x = 1 \;\Rightarrow\; y = 2$

or $x = -1 \;\Rightarrow\; y = -2$

but $x \neq 0$



So the curve rises on $(-\infty,-1]$ and $[1, \infty)$

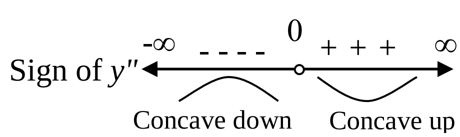
and it falls on $[-1,0)$ and $(0,1]$

and has relative max. at point $(-1,-2)$

and has relative min. at point $(1,2)$

3. Second derivative test:

$$y`` = \frac{2}{x^3} \quad \Rightarrow \quad y`` \neq 0$$



There is no inflection point because it is not defined at $x=0$.

The curve is concave up on $(0, \infty)$, and it is concave down on $(-\infty, 0)$

4. Asymptotes:

- Horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+1}{x} = \lim_{x\to-\infty} \frac{x^2/x + 1/x}{x/x} = \lim_{x\to-\infty} \frac{x+1/x}{1} = \frac{-\infty+0}{1} = -\infty$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+1}{x} = \lim_{x\to\infty} \frac{x^2/x + 1/x}{x/x} = \lim_{x\to\infty} \frac{x+1/x}{1} = \frac{\infty+0}{1} = \infty$$

So there is no horizontal asymptote.

- Vertical asymptotes:

To find a put the denominator equal zero.

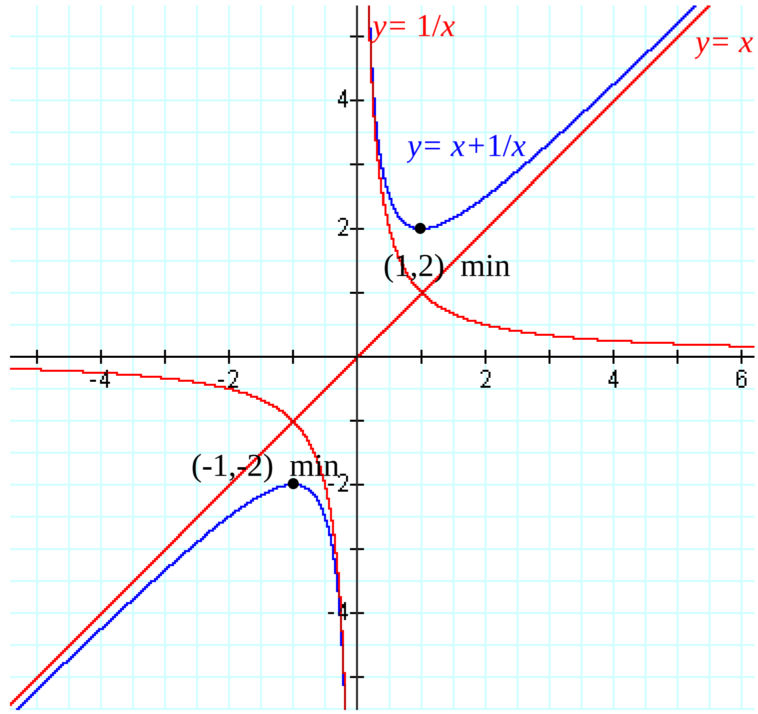
$x=0 \Rightarrow a=0$

$$\lim_{x\to a^-} f(x) = \lim_{x\to 0^-} \frac{x^2+1}{x} = \frac{0+1}{0^-} = -\infty$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to 0^+} \frac{x^2+1}{x} = \frac{0+1}{0^+} = +\infty$$

$\therefore x=0$ (y-axis) is vertical asymptote

- Oblique asymptotes: because of that the function is a rational function with nominator is one greater than the denominator. $\therefore y=x$ is an oblique asymptote



Homework:

1. Find asymptotes of the following curves then graph them.

(a) $y=\frac{1}{x-2}$ (b) $y=\frac{x-4}{x-5}$ (c) $y=\frac{x^2+4}{2x}$

(d) $y=\frac{x^2-4}{x-1}$ (e) $y=\frac{-x^2+2x-4}{x-1}$ (f) $y=\frac{x-1}{x^2(x-2)}$

(g) $y=\frac{8}{x^2+4}$ (h) $y=\frac{4x}{x^2+4}$ (i) $y=\frac{x^2+x-6}{x-1}$

(j) $y=\frac{x^2}{x^2-1}$ (k) $y=\frac{x^2-x+1}{x-1}$

2. Graph the following functions:

(a) $y=9x-x^2$ (b) $y=x^3-3x^2+3$ (c) $y=\frac{x^2-2x+2}{x-2}$

(d) $y = \frac{-x^2}{x+1}$ (e) $y = \frac{x^2}{x^2 - 1}$ (f) $y = \frac{x^2 - 1}{x}$

(g) $y = x^{1/3}$ (h) $y = \frac{8}{4 - x^2}$ (i) $y = |4 - x^2|$

(j) $y = \tan x + \sin x \quad 0 \le x \le 2\pi$ (k) $y = \sin|x| \quad -2\pi \le x \le 2\pi$

(l) $y = |\sin x| \quad -2\pi \le x \le 2\pi$

Note: When the function contains an absolute value, the function can be graphed as two parts.