Implicit Differentiation:

In some cases, it is difficult to solve $y=f(x)$, so to find $\frac{dy}{dx}$ for such cases, implicit differentiation will be use.

Examples: Find $\frac{dy}{dx}$ of the following:

1. $x^2 + y^2 = 1$

Sol.: $2x + 2y * \frac{dy}{dx} = 0 \Rightarrow 2y * \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

2. $2y = x^2 + 3xy^2$

Sol.: $2\frac{dy}{dx} = 2x + 3x(2y\frac{dy}{dx}) + 3y^2 \Rightarrow 2\frac{dy}{dx} - 6xy\frac{dy}{dx} = 2x + 3y^2 \Rightarrow \frac{dy}{dx}(2 - 6xy) = 2x + 3y^2$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + 3y^2}{2 - 6xy}$$

Tangent and Normal Lines:

Example: Find the tangent and normal to the curve $x^2 - xy + y^2 = 7$ at the point (-1,2).

Sol.: We first use the implicit differentiation to find $\frac{dy}{dx}$.

$$x^2 - xy + y^2 = 7$$

$$2x - (x\frac{dy}{dx} + y) + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(2y - x) = y - 2x$$

$$\therefore \frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

We then evaluate the derivative (slope of the curve) at $x=-1$ and $y=2$ to obtain:

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \frac{y - 2x}{2y - x} = \frac{2 - 2(-1)}{2(2) - (-1)} = \frac{2 + 2}{4 + 1} = \frac{4}{5}$$

So the tangent to the curve at the point (-1, 2) is:

$$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 2 = \frac{4}{5}(x - (-1))$$

$$\Rightarrow \quad y = \frac{4}{5}x + \frac{4}{5} + 2 \quad \Rightarrow \quad \therefore y = \frac{4}{5}x + \frac{14}{5}$$

And the normal to the curve at the point (-1, 2) is (slope of normal is (-1/m)):

$$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 2 = -\frac{5}{4}(x - (-1))$$

$$\Rightarrow \quad y = -\frac{5}{4}x - \frac{5}{4} + 2 \quad \Rightarrow \quad \therefore y = -\frac{5}{4}x + \frac{3}{4}$$

Example: Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$.

Sol.: to find $\frac{dy}{dx}$:

$$2x^3 - 3y^2 = 7 \quad \Rightarrow \quad 6x^2 - 6y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 6y\frac{dy}{dx} = 6x^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{6x^2}{6y} = \frac{x^2}{y} = y` \quad \text{where } y \neq 0$$

We now apply the Quotient Rule to find $\frac{d^2y}{dx^2}$ or (y'').

So $y`` = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{x^2}{y}) = \frac{y(2x) - x^2(\frac{dy}{dx})}{y^2} = \frac{2xy - x^2(\frac{x^2}{y})}{y^2}$

$$= \frac{2xy^2 - x^4}{y^3}$$

Homework:

syllabus

1. If $x^2y - xy^2 + x^2 + y^2 = 0$ find $y`$.

2. If $x^2 - xy + y^2 = 3$ find $y`$ and y''.

3. If $x^3y - xy^3 = 2$ find $y`$ and y'' at $x=1$.

4. Prove that the following curves are intersecting with a right angle:

$$5y - 2x + y^2 - x^2y = 0$$

$$2y + 5x + x^4 - x^3y^2 = 0$$ in the origin.

Hint: $y_1` * y_2` = -1$ in the origin $(0, 0)$.

The Chain Rule:

If $y = f(u)$; $u = g(x)$, and the derivatives $\frac{dy}{du}$ and $\frac{du}{dx}$ both exist then the composite function defined by $f(g(x))$ has a derivative given by:

$$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$$

Example 1: If $y = \sqrt{u^2 + 1}$; $u = \frac{1}{x} + x^2$, find $\frac{dy}{dx}$.

Sol.: $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$

$$\frac{dy}{du} = \frac{2u}{2\sqrt{u^2+1}} = \frac{u}{\sqrt{u^2+1}}; \quad \frac{du}{dx} = -\frac{1}{x^2} + 2x$$

$$\therefore \frac{dy}{dx} = \frac{u}{\sqrt{u^2+1}} * (2x - \frac{1}{x^2}) = \frac{(\frac{1}{x} + x^2)}{\sqrt{(\frac{1}{x} + x^2)^2 + 1}} * (2x - \frac{1}{x^2})$$

Another solution:

Find $y_o u = y(u(x)) = \sqrt{(\frac{1}{x} + x^2)^2 + 1}$

$$\therefore \frac{dy}{dx}=\frac{2(\frac{1}{x}+x^2)*(-\frac{1}{x^2}+2x)}{2\sqrt{(\frac{1}{x}+x^2)^2+1}}=\frac{(\frac{1}{x}+x^2)*(2x-\frac{1}{x^2})}{\sqrt{(\frac{1}{x}+x^2)^2+1}}$$

Example 2: If $y=(3x^2-7x+1)^5$, use the chain rule to find $\frac{dy}{dx}$.

Sol.: We may express *y* as a composite function of *x* by letting:

$$y=u^5 \quad \text{and} \quad u=3x^2-7x+1$$

So, $\frac{dy}{dx}=\frac{dy}{du}*\frac{du}{dx}=5u^4*(6x-7)=5(3x^2-7x+1)^4(6x-7)$

Homework: Find $\frac{dy}{dx}$ at *x* = -1 if $y=u^3+5u-4$ and $u=x^2+x$.

Derivative of Parametric Equations:

If $y=f(t)$ and $x=g(t)$, and the derivatives $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both exist, then:

$$\frac{dy}{dx}=y`=\frac{dy/dt}{dx/dt}$$

and

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\frac{d}{dx}y)=\frac{dy`}{dx}=\frac{dy`/dt}{dx/dt}$$

Example 1: Find $\frac{dy}{dx}$, if $y=t^2-1$ and $x=2t+3$.

Sol.: $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$

So $\frac{dy}{dt}=2t$ and $\frac{dx}{dt}=2$ ➔ $\therefore \frac{dy}{dx}=\frac{2t}{2}=t=\frac{x-3}{2}$

Another solution:

From $x = 2t + 3$ find $t = \frac{x-3}{2}$

Then: $y = \left(\frac{x-3}{2}\right)^2 - 1$

$\therefore \frac{dy}{dx} = 2\left(\frac{x-3}{2}\right) * \frac{1}{2} = \frac{x-3}{2}$

Example 2: Find $\frac{d^2y}{dx^2}$, if $x = t - t^2$ and $y = t - t^3$.

Sol.: $\frac{dx}{dt} = 1 - 2t$ and $\frac{dy}{dt} = 1 - 3t^2$

$\therefore \frac{dy}{dx} = y` = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$

And $\frac{d^2y}{dx^2} = \frac{dy`/dt}{dx/dt}$

$\frac{dy`}{dt} = \frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^2} = \frac{6t^2 + 6t + 2}{(1-2t)^2}$

$\therefore \frac{d^2y}{dx^2} = \frac{dy`/dt}{dx/dt} = \frac{6t^2 + 6t + 2}{(1-2t)^3}$

Homework:

1. Find $\frac{dr}{dt}$ if $r = \sqrt{s+1}$ and $s = 16t^2 - 20t$.
2. Find $\frac{d^2y}{dt^2}$ if $y = x^2 + 3x - 7$ and $x = 2t + 1$.
3. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x = \frac{t}{1+t}$ and $y = \frac{t^2}{1+t}$.

Derivative of Trigonometric Functions:

1. The derivative of $y = \sin x$ is the limit:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\sin x\cos\Delta x+\cos x\sin\Delta x-\sin x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin x(\cos\Delta x-1)}{\Delta x}+\lim_{\Delta x\to 0}\frac{\cos x\sin\Delta x}{\Delta x}$$
$$=\sin x\lim_{\Delta x\to 0}\frac{(\cos\Delta x-1)}{\Delta x}+\cos x\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}=\sin x(0)+\cos x(1)=\cos x$$
$$\therefore\frac{d}{dx}\sin x=\cos x$$

2. The derivative of $y=\cos x$ is the limit:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\cos x\cos\Delta x-\sin x\sin\Delta x-\cos x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cos x(\cos\Delta x-1)}{\Delta x}-\lim_{\Delta x\to 0}\frac{\sin x\sin\Delta x}{\Delta x}$$
$$=\cos x\lim_{\Delta x\to 0}\frac{(\cos\Delta x-1)}{\Delta x}-\sin x\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}=\cos x(0)-\sin x(1)=-\sin x$$
$$\therefore\frac{d}{dx}\cos x=-\sin x$$

Example: Find $\frac{dy}{dx}$ if $y=\sec x$

Sol.: $y=\sec x=\frac{1}{\cos x}$

$$\therefore\frac{dy}{dx}=\frac{\cos x(0)-(-\sin x)}{\cos^2 x}=\frac{\sin x}{\cos^2 x}=\frac{1}{\cos x}.\frac{\sin x}{\cos x}=\sec x.\tan x$$
$$\therefore\frac{d}{dx}\sec x=\sec x.\tan x$$

In general: u is function of x

1. $\frac{d}{dx}\sin u=\cos u*\frac{du}{dx}$
2. $\frac{d}{dx}\cos u=-\sin u*\frac{du}{dx}$
3. $\frac{d}{dx}\tan u=\sec^2 u*\frac{du}{dx}$
4. $\frac{d}{dx}\cot u=-\csc^2 u*\frac{du}{dx}$
5. $\frac{d}{dx}\sec u=\sec u.\tan u*\frac{du}{dx}$
6. $\frac{d}{dx}\csc u=-\csc u.\cot u*\frac{du}{dx}$

Example: Find $\frac{dy}{dx}$ of the following functions:

1. $y = x^2 - \sin x$

Sol.: $\frac{dy}{dx} = 2x - \cos x$

2. $y = x^2 \sin x$

Sol.: $\frac{dy}{dx} = x^2 \cos + 2x \sin x$

3. $y = \frac{\sin x}{x}$

Sol.: $\frac{dy}{dx} = \frac{x \cos x - \sin x}{x^2}$

4. $y = \sin x \cos x$

Sol.: $\frac{dy}{dx} = \sin x.(-\sin x) + \cos x.\cos x = -\sin^2 x + \cos^2 x = \cos^2 x - \sin^2 x = \cos 2x$

or $y = \sin x \cos x = \frac{\sin 2x}{2}$

$\therefore \frac{dy}{dx} = \frac{\cos 2x}{2} * 2 = \cos 2x$

5. $y = \frac{\cos x}{1 - \sin x}$

Sol.: $\frac{dy}{dx} = \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$

$= \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1 - \sin x}{(1-\sin x)^2} = \frac{1}{1 - \sin x}$

6. $y = \cos^2 3x$

Sol.: $\frac{dy}{dx} = 2\cos 3x * (-\sin 3x) * 3 = -3(2\cos 3x \sin 3x) = -3\sin 6x$

7. $y = \sin(1 + \tan 2x)$

Sol.: $\frac{dy}{dx}=\cos(1+\tan 2x)\sec^2 2x*2=2\sec^2 2x\cos(1+\tan 2x)$

8. $y=\tan(\frac{1}{x})$

Sol.: $\frac{dy}{dx}=\sec^2(\frac{1}{x})*(-\frac{1}{x^2})=-\frac{1}{x^2}\sec^2(\frac{1}{x})$

9. $xy+\sin y=5x$

Sol.: $xy`+y+\cos y*y`=5 \quad \Rightarrow \quad y`(x+\cos y)=5-y$

$\Rightarrow \quad y`=\frac{5-y}{x+\cos y}$

10. $y=\sec^2 5x$

Sol.: $\frac{dy}{dx}=2\sec 5x.\sec 5x.\tan 5x*5=10\sec^2 5x.\tan 5x$

11. $x=\sin y-\sqrt{y}$

Sol.: $\frac{dx}{dy}=\cos y-\frac{1}{2\sqrt{y}}=\frac{2\sqrt{y}\cos y-1}{2\sqrt{y}}$

$\therefore \frac{dy}{dx}=1/(\frac{dx}{dy})=\frac{2\sqrt{y}}{2\sqrt{y}\cos y-1}$

or by implicit differentiation:

$1=\cos y.y`-\frac{1}{2\sqrt{y}}y` \quad \Rightarrow \quad 1=\frac{2\sqrt{y}\cos y.y`-y`}{2\sqrt{y}}$

$\Rightarrow \quad 1=\frac{y`(2\sqrt{y}\cos y-1)}{2\sqrt{y}} \quad \Rightarrow \quad y`=\frac{2\sqrt{y}}{2\sqrt{y}\cos y-1}$

12. $y=\sqrt{1+\cos 2x}$

Sol.: $\frac{dy}{dx}=\frac{1}{2}(1+\cos 2x)^{\frac{-1}{2}}*(-\sin 2x)*2=\frac{-\sin 2x}{\sqrt{1+\cos 2x}}$

13. $y=x\sin x+\cos x$

Sol.: $\frac{dy}{dx} = x\cos x + \sin x - \sin x = x\cos x$

3. $y = \sqrt{\frac{1+\cos 2x}{2}}$

Sol.: $y = \sqrt{\frac{1+\cos 2x}{2}} = \sqrt{\cos^2 x} = |\cos x| = \begin{cases} \cos x & if \quad \cos x \geq 0 \\ -\cos x & if \quad \cos x < 0 \end{cases}$

$$\therefore \frac{dy}{dx} = \begin{cases} -\sin x & if \quad \cos x \geq 0 \\ \sin x & if \quad \cos x < 0 \end{cases}$$

Example: If $y = \sec x$, prove that $y`` + y = 2y^3$.

Sol.: $y = \sec x$

$y` = \sec x.\tan x$

$y`` = \sec x.\sec^2 x + \tan x.\sec x.\tan x$

$= \sec^3 x + \sec x.\tan^2 x$

$= \sec^3 x + \sec x.(\sec^2 x - 1)$

$= \sec^3 x + \sec^3 x - \sec x$

$= 2\sec^3 x - \sec x$

$= 2y^3 - y$

$\therefore y`` + y = 2y^3$ o.k.

Homework:

1. If $y = \tan x$, prove that $y``` = 2(y^2 + 1)(3y^2 + 1)$.
2. If $x = \sec t$ and $y = \tan t$, evaluate at $t = \frac{\pi}{4}$ of the following:

   $(a)\ \frac{dy}{dx}$ $(b)\ \frac{d^2y}{dx^2}$

3. Find $y`$ for $y = \cos(\sin^2 x)$

4. If $x = y\sqrt{1-y^2}$, find $\frac{dy}{dx}$.

5. If $y = x^2 - 4x$ and $x = \sqrt{2t^2+1}$; find $\frac{dy}{dt}$ when $t = \sqrt{2}$.

6. Find $y`$ and $y``$ for $y = \sin(x+y)$.

7. Find $y`$ for $\sin x + \sin y = 1$.

8. Find $y`$ for $x\cos y = \sin(x+y)$.