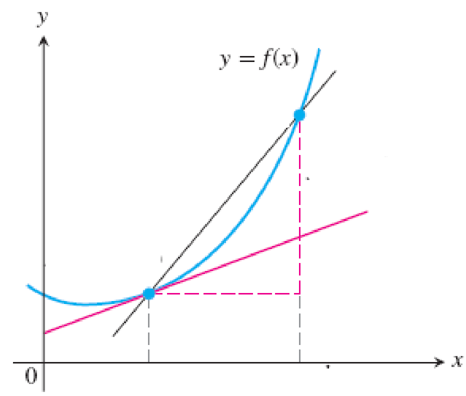
DERIVATIVES

Derivatives: are the functions which are used to measure rates at which things change. We define derivatives as limiting values of average change, just we define slope of curves as limiting values of slopes of secants.



If $\quad y = f(x)$

$\therefore \Delta y = f(x+\Delta x) - f(x)$

So**,** slope of secant $PQ = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

As $Q \to P$ then slope of secant PQ will equal to slope of tangent of the curve $f(x)$ at P and $\Delta x \to 0$

$\therefore \lim\limits_{Q \to P}$ slope of secant $PQ = \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} =$ slope of tangent of the curve $f(x)$ at P.

And this is called the definition of derivative of the function $f(x)$ and this denoted by $y`$, $\dfrac{dy}{dx}$, $\dfrac{d}{dx} f(x)$, $D_x f(x)$ and $f`(x)$.

$$\therefore f`(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

We consider that the derivative is found if the limit exists and finite at a certain point.

Example1: Find the derivative of the function $f(x) = x^2$ using the definition of derivative.

Sol.: $\dfrac{dy}{dx} = f`(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{(x+\Delta x)^2 - x^2}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x.\Delta x + \Delta x^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0}(2x+\Delta x) = 2x$$

Example2: Find the derivative of the function $f(x)=\sqrt{x}$ using the definition of derivative.

Sol.: $$\frac{dy}{dx}=f`(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}*\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$=\lim_{\Delta x\to 0}\frac{x+\Delta x-x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}==\lim_{\Delta x\to 0}\frac{1}{(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$=\frac{1}{(\sqrt{x}+\sqrt{x})}=\frac{1}{2\sqrt{x}}$$

Homework:

1. Using the definition of derivative, find the derivative of the following functions:

   $(a)\ y=x\sqrt{x^2-2}$ $\quad$ $(b)\ y=\frac{x^2}{(x-1)^2}$ $\quad$ $(c)\ y=\frac{1}{2\sqrt{x}}$

2. Using the definition of derivative, prove the following:

   $(a)\ f`(x)=\frac{1}{3\sqrt[3]{x}}$ $\quad$ if $f(x)=\sqrt[3]{x}$

   $(b)\ y`=\sec^2 x$ $\quad$ if $y=\tan x$

Laws of Derivatives:

1. $\frac{d}{dx}c=0$ $\quad$ where c is constant.

2. $\frac{d}{dx}x^n=n.x^{n-1}$

3. If $U\ and\ V$ are two functions of x then:

   $(a)\ \frac{d}{dx}(c*U)=c*\frac{dU}{dx}$ $\quad$ where c is constant.

   $(b)\ \frac{d}{dx}(U\mp V)=\frac{dU}{dx}\mp\frac{dV}{dx}$

   $(c)\ \frac{d}{dx}(U*V)=U\frac{dV}{dx}+V\frac{dU}{dx}$

(d) $\frac{d}{dx}(U^n) = nU^{n-1} * \frac{dU}{dx}$

(e) $\frac{d}{dx}(\frac{U}{V}) = \frac{V\frac{dU}{dx} - U\frac{dV}{dx}}{V^2}$

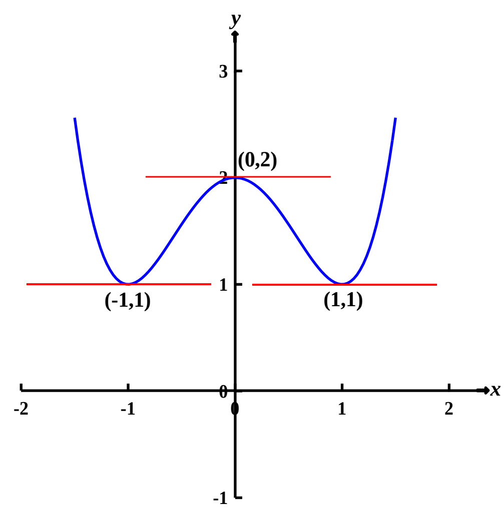
Example1: If $y = x^3 + 7x^2 - 5x + 4$, find $\frac{dy}{dx}$?

Sol.: $\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(7x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(4)$

$$= 3x^2 + 2*7x - 5 + 0 = 3x^2 + 14x - 5$$

Example2: Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangent? If so, where?

Sol.: The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points, we should



1. Calculate dy/dx:

$$\frac{dy}{dx} = 4x^3 - 2(2x) = 4x^3 - 4x$$

2. Put $\frac{dy}{dx} = 0 \qquad \Rightarrow 4x^3 - 4x = 0$

3. Solve the equation $\frac{dy}{dx} = 0$ for x.:

$$4x^3 - 4x = 0 \quad \Rightarrow \quad 4x(x^2 - 1) = 0$$

either $\quad 4x = 0 \quad \Rightarrow \quad x = 0$

or $\quad x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1$

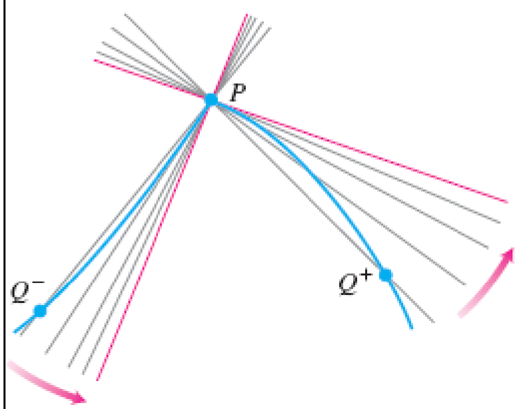
So the curve has horizontal tangents at x=0, x=-1 and x=1

The corresponding points on the curve (calculated from the equation $y = x^4 - 2x^2 + 2$) are (0,2), (-1,1) and (1,1).

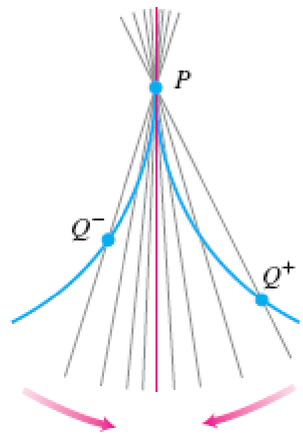
When does a Function *not* Have a Derivative at a Point?

A function has a derivative at a point x_o if the slopes of the secant lines through $P(x_o, f(x_o))$ and a nearby point Q on the graph approach a limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist. Thus differentiability is a "smoothness" condition on the graph of f. A function whose graph is otherwise smooth will fail to have a derivative at a point for several reasons, such as at points where the graph has

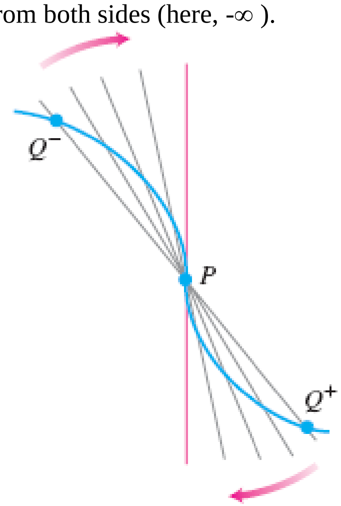
1. a *corner*, where the one-sided derivatives differ.



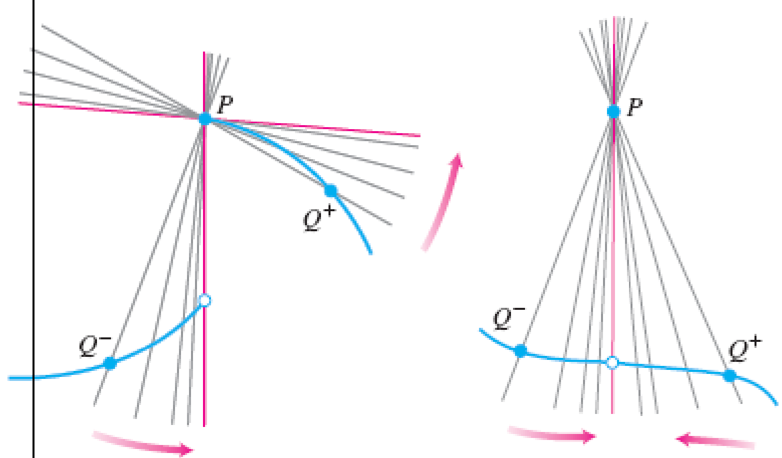
2. a *cusp*, where the slope of *PQ* approaches ∞ from one side and -∞ from the other.



3. a *vertical tangent*, where the slope of *PQ* approaches ∞ from both sides or approaches - ∞ from both sides (here, -∞).



4. a *discontinuity*.



Example 1: Show that the function $y=|x|$ is differentiable on $(-\infty,0)$ and $(0,\infty)$ but has no derivative at $x=0$.

Sol. To the right of the origin ($x>0$),

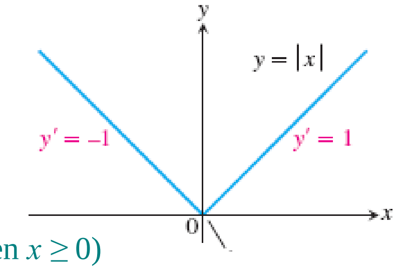
$$\frac{d}{dx}\left(|x|\right)=\frac{d}{dx}(x)=1$$

To the left of the origin ($x < 0$),

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = -1$$

There can be no derivative at the origin because the one-sided derivatives differ there:

Or another solution: Right-hand derivative of $|x|$ at zero



$$\frac{dy}{dx} = \lim_{\Delta x \to 0^+} \frac{f(x+\Delta x) - f(x)}{\Delta x} == \lim_{\Delta x \to 0^+} \frac{|x+\Delta x| - |x|}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0^+} \frac{(x+\Delta x) - (x)}{\Delta x} \qquad (|x| = +x \text{ when } x \geq 0)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0^+} 1 = 1$$

And left-hand derivative of $|x|$ at zero

$$\frac{dy}{dx} = \lim_{\Delta x \to 0^-} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{|x+\Delta x| - |x|}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0^-} \frac{-(x+\Delta x) - (-x)}{\Delta x} \qquad (|x| = -x \text{ when } x < 0)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0^-} \frac{-x - \Delta x + x}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

The function $y=|x|$ is not differentiable at the origin where the graph has a "corner".

***Example* 2**: Show that the function $y = \sqrt{x}$ is not differentiable at $x = 0$.

Sol. We apply the definition to examine if the derivative exists at $x = 0$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0^+} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0^+} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} * \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0^+} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0^+} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

And at $x = 0$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{0}} = \infty$$

Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $(x, \sqrt{x})$ on a graph of $y = \sqrt{x}$ approach ∞ the graph has a *vertical tangent* at the origin.

Second and Higher Order Derivative:

The derivative: $y` = \frac{dy}{dx}$ is the first derivative of y with respect to x.

The first derivative may also be a differentiable function of x. If so its derivative:

$y`` = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}$ is the second derivative of y with respect to x.

If $y``$ is also a differentiable function of x, its derivative:

$y``` = \frac{d}{dx}(y``) = \frac{d}{dx}(\frac{d^2y}{dx^2}) = \frac{d^3y}{dx^3}$ is the third derivative of y with respect to x.

The names continue as you imagine with

$y^n = \frac{d}{dx}(y^{n-1}) = \frac{d}{dx}(\frac{d^{n-1}y}{dx^{n-1}}) = \frac{d^ny}{dx^n}$ is the n^{th} derivative of y with respect to x for any positive integer n.

Example: The first four derivatives of $y = x^3 - 3x^2 + 2$ are:

First derivative: $y` = 3x^2 - 6x$,

Second derivative: $y'' = 6x - 6$,

Third derivative: $y''' = 6$

and Fourth derivative: $y'''' = 0$

The function has derivatives of all orders, but the fifth and subsequent order derivatives are all zero.

syllabus

Homework: Find (y` or $\frac{dy}{dx}$) and (y`` or $\frac{d^2y}{dx^2}$) for the following functions:

1. $y = x\sqrt{x^2 - 2}$
2. $y = \frac{x^2}{(x-1)^2}$
3. $y = (x^2 + 1)(x^3 + 3)$
4. $y = (x^2 + 3x + 1)^5$
5. $y = (x^2 + 1)^3 (x^3 - 1)^2$
6. $y = \left(\frac{2x-1}{x+7}\right)^3$