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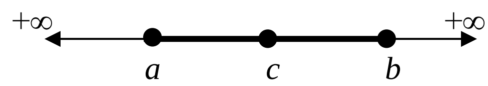
Continuous Functions:

DEFINITION:

- Continuity at interior points:

  A function $y=f(x)$ is continuous at an interior point c of its domain if:

$$\lim_{x\to c} f(x) = f(c)$$



- Continuity at end-points:

  A function $y= f(x)$ is continuous at a left end-point a of its domain if:

$$\lim_{x\to a^+} f(x) = f(a)$$

  A function $y=f(x)$ is continuous at a right end-point b of its domain if:

$$\lim_{x\to b^-} f(x) = f(b)$$

Continuous Functions:

A function is continuous if it is continuous at each point of its domain.

Discontinuity at a point:

If a function $f(x)$ is not continuous at a point c, we say that $f(x)$ is discontinuous at c and call c a point of discontinuity of $f(x)$.

The Continuity Test

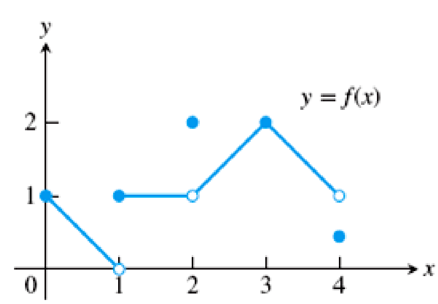
The function $y=f(x)$ is continuous at $x=c$ if and only if the following statements are true:-

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x\to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).

Example1: Discuss the continuity conditions of the function $f(x)$ which shown in figure at $x=0$, $x=1$, $x=2$, $x=3$, $x=1.5$ and $x=4$.

-at $x=0$ (left end-point)

$$f(0)=1$$



$\lim_{x\to 0^+} f(x) = 1$

$\therefore \lim_{x\to 0^+} f(x) = f(0) = 1$

So it is continuous at the left end-point (x=0).

-at x=1 (interior point)

$f(1) = 1$

$\lim_{x\to 1^-} f(x) = 0$

$\lim_{x\to 1^+} f(x) = 1$

$\therefore \lim_{x\to 1} f(x)$ does not exist, because the right-hand and left-hand limits are not equal.

So it is discontinuous at x=1.

-at x=2 (interior point)

$f(2) = 2$

$\lim_{x\to 2^-} f(x) = 1$

$\lim_{x\to 2^+} f(x) = 1$

$\therefore \lim_{x\to 2} f(x) = 1$

$\therefore \lim_{x\to 2} f(x) \neq f(2)$

So it is discontinuous at x=2.

-at x=3 (interior point)

$f(3) = 2$

$\lim_{x\to 3^-} f(x) = 2$

$\lim_{x\to 3^+} f(x) = 2$

$\therefore \lim_{x\to 3} f(x) = 2$

$\therefore \lim_{x\to 3} f(x) = f(3) = 2$

So it is continuous at x=3.

-at x=1.5 (interior point)

$f(1.5) = 1$

$$\lim_{x\to1.5^-} f(x) = 1$$

$$\lim_{x\to1.5^+} f(x) = 1$$

$$\therefore \lim_{x\to1.5} f(x) = 1$$

$$\therefore \lim_{x\to1.5} f(x) = f(1.5) = 1$$

So it is continuous at x=1.5.

-at x=4 (right end-point)

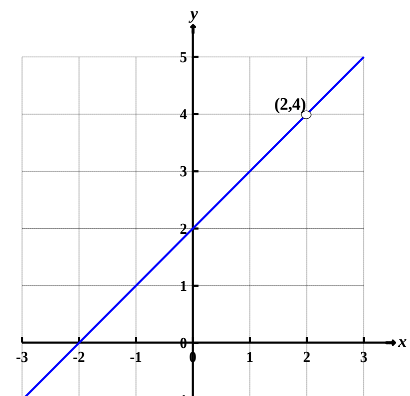
$f(4) = 0.5$

$$\lim_{x\to4^-} f(x) = 1$$

$$\therefore \lim_{x\to4^-} f(x) \neq f(4)$$

So it is discontinuous at right-end point (x=4).

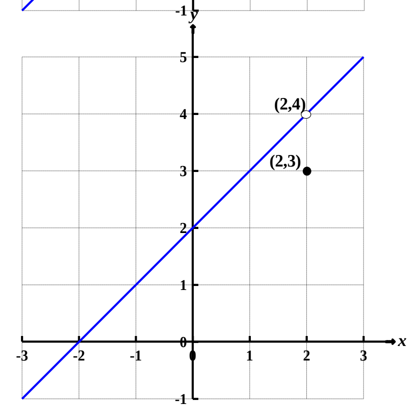
Example2: Determine weather the following functions are continuous at x=2?



1. $f(x) = \dfrac{x^2-4}{x-2}$

Sol.: $f(2)$ is not found ($2 \notin D_f$)

So the function is discontinuous at x=2.



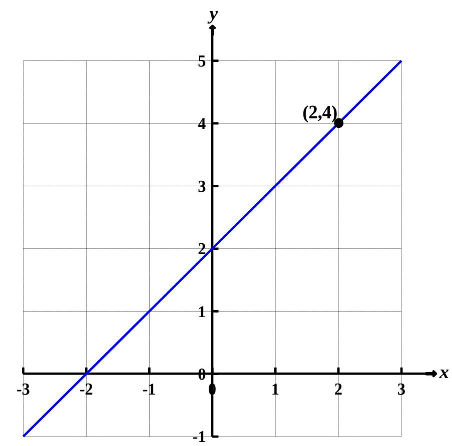
2. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$

Sol.: $f(2)$=3

$$\lim_{x\to2} \frac{x^2-4}{x-2} = \lim_{x\to2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to2} x+2 = 2+2 = 4$$

$$\therefore\ f(2) \neq \lim_{x\to 2} f(x)$$

So the function is discontinuous at x=2.



3. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$

Sol.: $f(2)=4$

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

$$\therefore\ f(2) = \lim_{x\to 2} f(x)$$

So the function is continuous at x=2.

Example1: Test the continuity of the following function at x=1:

$$f(x) = \begin{cases} x^2 & x < 1 \\ \dfrac{x}{2} & x \geq 1 \end{cases}$$

Sol.: $f(1) = \dfrac{1}{2}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = (1)^2 = 1 \qquad , \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} \frac{x}{2} = \frac{1}{2}$$

$\lim_{x\to 1} f(x)$ is not found (the left-hand and right-hand limits do not equal).

So the function is discontinuous at x=1.

Algebraic properties of continuous functions:

Theorem 1: If the functions f and g are continuous at c then:

a) $f+g$, $f-g$ and $f*g$ are continuous at $x=c$.

b) $\dfrac{f}{g}$ is continuous if $g(c) \neq 0$ and is discontinuous $x=c$ if $g(c)=0$.

Theorem 2: If f is continuous at c and g is continuous at $f(c)$ then the composite $g_o f$ is continuous at c.

Theorem 3: Polynomials are continuous at every point.

$$[a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots]$$

Examples of continuous functions:

1. The function $y = \frac{1}{x}$ is continuous at every value of x except x=0.
2. The greatest integer function y=[x] is discontinuous at every integer.
3. The sine and cosine functions are continuous at every value of x.
4. Polynomials are continuous at every value of x.

   For polynomials: $\lim_{x \to c} f(x) = f(c)$
5. Rational functions are continuous wherever they are defined.
6. The function y=|x| is continuous at every value of x.

Homework:

1. Test the continuity of the following functions at given points?

(a) $$f(x) = \begin{cases} x^2 - 1 & -1 \le x < 0 \\ 2x & 0 \le x < 1 \\ 1 & x = 1 \\ -2x+4 & 1 < x < 2 \\ 0 & 2 \le x \le 3 \end{cases}$$

at $x = -1$, $x = 0$, $x = 1$, $x = 2$ and $x = 3$.

(b) $f(x) = \frac{x^2 - 4}{(x+3)(x-2)}$ at $x = 2$ and $x = -3$.

(c) $$\begin{cases} \frac{x^3 - 27}{x^2 - 9} \end{cases}$$

at $x = 3$ and $x = -3$

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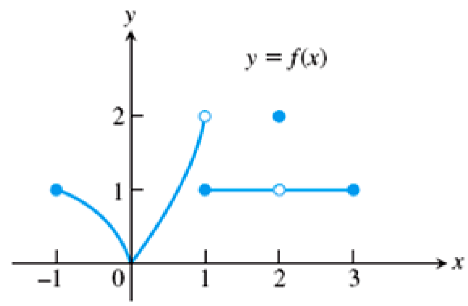
(d) $f(x)=\dfrac{x^2+x-2}{(x-1)^2}$ at $x=1$

2. Which the following statements are true or false of the function graphed here.



(a) $\lim\limits_{x\to -1^+} f(x)=1$

(b) $\lim\limits_{x\to 2} f(x)$ does not exist

(c) $\lim\limits_{x\to 2} f(x)=2$

(d) $\lim\limits_{x\to 1^-} f(x)=2$

(e) $\lim\limits_{x\to -1^+} f(x)=1$

(f) $\lim\limits_{x\to 1} f(x)$ does not exist

(g) $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)$

(h) $\lim\limits_{x\to -1^-} f(x)=1$

(i) $\lim\limits_{x\to 3^+} f(x)$ does not exist

(j) $\lim\limits_{x\to c} f(x)$ exists at every c in (-1,1)

(k) $\lim\limits_{x\to c} f(x)$ exists at every c in (1,3)

3. Find the limits of the following functions:

(a) $\lim\limits_{x\to 0^+}\dfrac{|x|}{x}$

(b) $\lim\limits_{x\to 0^-}\dfrac{|x|}{x}$

(c) $\lim\limits_{x\to 0}[\cos x]$

(d) $\lim\limits_{x\to 4^+}[[x]]$

(e) $\lim\limits_{x\to 0^+}\dfrac{\sqrt{x}}{\sqrt{4+\sqrt{x}}-1}$

(f) $\lim\limits_{n\to\infty}\dfrac{n^2-5n+\sin n}{4n^2+7n+6}$

(g) $\lim\limits_{n\to\infty}\sqrt{n^2+n}-\sqrt{n^2+10}$

(h) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x+\cos x}$

(i) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{2x+5}$

(j) $\lim\limits_{x\to\infty}\dfrac{1+\sin x}{x}$

(k) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x}$

(l) $\lim\limits_{n\to\infty}\dfrac{2n^2-5n-3}{n^3+2n}$

(m) $\lim\limits_{x\to 3^+}\dfrac{[x]}{x}$

(n) $\lim\limits_{x\to 3^-}\dfrac{[x]}{x}$

(o) $\lim\limits_{x\to 4^+} x-[x]$

4. Use the sandwich theorem to find the following limits:

(a) $\lim\limits_{x\to\infty}\dfrac{\sin x}{\sqrt{x}}$

(b) $\lim\limits_{x\to\infty}\dfrac{\cos x}{\sqrt{x}}$