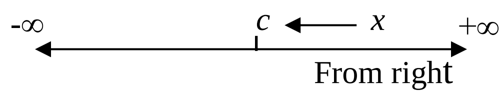
Right-hand limits and left-hand limits

The notation for the *right-hand* limit is

$\lim_{x\to c^+} f(x)$ "The limit of $f(x)$ as x approaches c from the *right*"

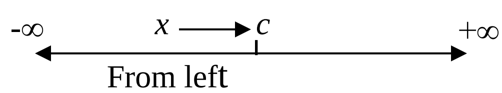
The (+) is there to say that x approaches c through values *greater than* c on the line numbers.



The notation for the *left-hand* limit is

$\lim_{x\to c^-} f(x)$ "The limit of $f(x)$ as x approaches c from the *left*"

The (-) is there to say that x approaches c through values *less than* c on the line numbers.

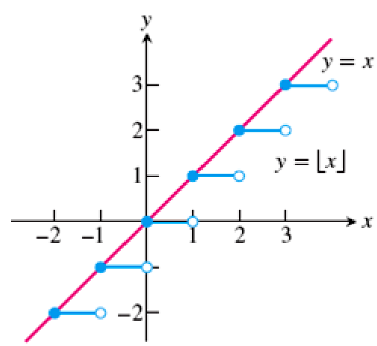


Examples: The greatest integer function $f(x) = [x]$ has difference right-hand and left-hand limits at each integer. As we see in figure:

1. $\lim_{x\to 2^+}[x] = 2$ but $\lim_{x\to 2^-}[x] = 1$
2. $\lim_{x\to -1^+}[x] = -1$ but $\lim_{x\to -1^-}[x] = -2$

While

3. $\lim_{x\to 1.5^+}[x] = 1$ but $\lim_{x\to 1.5^-}[x] = 1$



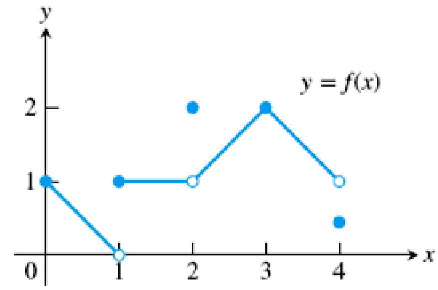
One sided vs. two sided limits:

DEFINITION:

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbol:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = L \text{ and } \lim_{x\to c^-} f(x) = L$$

Example: Discuss the limit properties of the function $f(x)$ which shown in figure.



- at x=0 $\lim_{x\to 0^+} f(x) = 1$

  $\lim_{x\to 0^-} f(x)$ does not exist (because the function is not defined to the left of x=0)

- at x=1 $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$

  $\lim_{x\to 1^+} f(x) = 1$

  $\lim_{x\to 1} f(x)$ does not exist, because the right-hand and left-hand limits are not equal.

- at x=2 $\lim_{x\to 2^-} f(x) = 1$

  $\lim_{x\to 2^+} f(x) = 1$

  $\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$

- at x=3 $\lim_{x\to 3^-} f(x) = 2$

  $\lim_{x\to 3^+} f(x) = 2$

  $\lim_{x\to 3} f(x) = f(3) = 2$

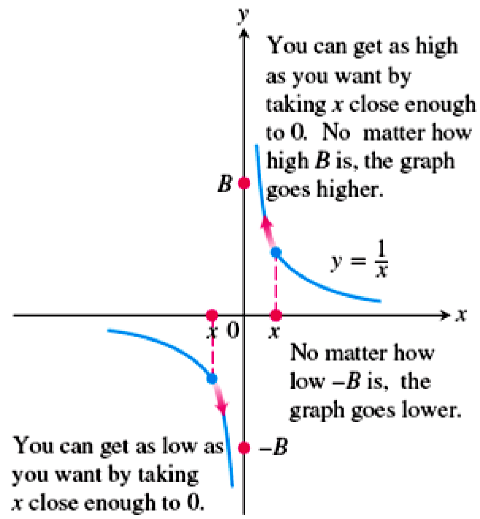
- at x=4 $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) = 0.5$

  $\lim_{x\to 4^+} f(x)$ does not exist, because the function is not defined to the right of x=4.

Limit Involving Infinity:

It means that the limits include $x \to +\infty$ or $x \to -\infty$ and $\lim f(x)=\infty$ or $\lim f(x)=-\infty$

Let $y = \frac{1}{x}$ then



1. $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ the limit does not exit.

2. $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ the limit does not exit.

3. $\lim_{x\to +\infty} \frac{1}{x} = 0$

4. $\lim_{x\to -\infty} \frac{1}{x} = 0$

Examples: Find the limits of the following:

1. $\lim_{x\to\infty} (5+\frac{1}{x}) = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x} = 5+0 = 5$

2. $\lim_{x\to\infty} \frac{x}{7x+4} = \lim_{x\to\infty} \frac{x/x}{7x/x+4/x} = \lim_{x\to\infty} \frac{1}{7+4/x} = \frac{1}{7+0} = \frac{1}{7}$

Note: In rational functions when *x* approaches infinity divide both the numerator and denominator by the largest bower of *x* in the denominator.

3. $\lim_{x\to\infty} \frac{2x^2-x+3}{3x^2+5} = \lim_{x\to\infty} \frac{2x^2/x^2 - x/x^2 + 3/x^2}{3x^2/x^2+5/x^2} = \lim_{x\to\infty} \frac{2-1/x+3/x^2}{3+5/x^2} = \frac{2-0+0}{3+0} = \frac{2}{3}$

4. $\lim_{x\to\infty} \frac{4x^2-3}{3x} = \lim_{x\to\infty} \frac{4x^2/x - 3/x}{3x/x} = \lim_{x\to\infty} \frac{4x-3/x}{3} = \frac{4*(\infty)}{3} = \infty$ the limit does not exit.

5. $\lim_{x\to\infty} \frac{5x+3}{2x^2-1} = \lim_{x\to\infty} \frac{5x/x^2+3/x^2}{2x^2/x^2-1/x^2} = \lim_{x\to\infty} \frac{5/x+3/x^2}{2-1/x} = \frac{0-0}{2-0} = \frac{0}{2} = 0$

Summery for Rational Functions

a) $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0$ if deg(f) < deg(g)

b) $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)}$ is finite if deg(f)=deg(g)

c) $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)}$ is infinite if deg(f) > deg(g)

6. $\lim_{x\to\infty} \frac{\sin x}{x}$

Sol.: Remember $-1 \le \sin x \le 1$ divide the inequality by x yield

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$\lim_{x\to\infty} -\frac{1}{x} = 0$ and $\lim_{x\to\infty} \frac{1}{x} = 0$

$\therefore \lim_{x\to\infty} \frac{\sin x}{x} = 0$ (sandwich theorem)

7. $\lim_{x\to\infty} x \sin \frac{1}{x}$

Sol.: Let $x = \frac{1}{z} \Rightarrow z = \frac{1}{x}$

When $x \to \infty \Rightarrow z \to 0$

$\therefore \lim_{x\to 0} \frac{1}{z} \sin z = 1$

8. $\lim_{x\to\infty} \sqrt{x^2+6x+1} - \sqrt{x^2+x} = (\infty - \infty)$

Sol.: $\lim_{x\to\infty} \sqrt{x^2+6x+1} - \sqrt{x^2+x} * \frac{\sqrt{x^2+6x+1} + \sqrt{x^2+x}}{\sqrt{x^2+6x+1} + \sqrt{x^2+x}}$

$$= \lim_{x\to\infty} \frac{(x^2+6x+1)-(x^2+x)}{\sqrt{x^2+6x+1}+\sqrt{x^2+x}} = \lim_{x\to\infty} \frac{x^2+6x+1-x^2-x}{\sqrt{x^2+6x+1}+\sqrt{x^2+x}} = \lim_{x\to\infty} \frac{5x+1}{\sqrt{x^2+6x+1}+\sqrt{x^2+x}}$$

$$= \lim_{x\to\infty} \frac{5x/x+1/x}{\sqrt{x^2/x^2+6x/x^2+1/x^2}+\sqrt{x^2/x^2+x/x^2}} = \frac{5+0}{\sqrt{1+0+0}+\sqrt{1+0}} = \frac{5}{1+1} = \frac{5}{2} = 2.5$$

Homework: Find the limits of the following:

[*syllabus*]()

1. $\lim_{x\to\infty} \frac{2x + x\sin 3x}{5x^2 - 2x + 1}$

2. $\lim_{x\to\infty} \sqrt{x+\sqrt{x}} - \sqrt{x}$

3. $\lim_{x\to\infty} \frac{(2x-1)^5}{(3x^2+2x-7)(x^3-9x)}$

4. $\lim_{x\to\infty} (2 + \frac{\sin x}{x})$

5. $\lim_{x\to\infty} \frac{\cos(1/x)}{(1/x)}$

6. $\lim_{x\to\infty} \frac{x}{|x|}$

7. $\lim_{x\to 0} \frac{|x|}{x}$