LIMITS AND CONTINUTY

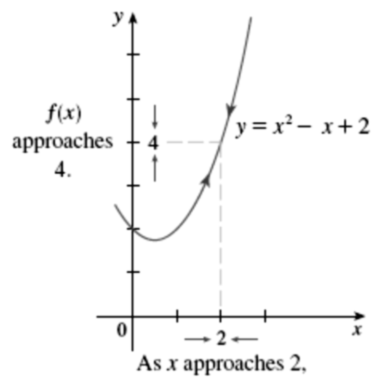
The most basic use of limits is to describe how a function behavior as the independent variable approaches a given value.

Example: Let us examine the behavior of the function $f(x)=x^2-x+2$ for a value of x closer and closer to 2.

x	1	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05
$f(x)$	2	2.75	3.71	3.852	3.97	3.985	3.997	///	4.003	4.015	4.03	4.152

Left side → ← Right side

It is evident from the graph and the table that the value of $f(x)$ gets closer and closer to 4 as value of x is selected closer and closer to 2 on either left or the right side of 2. We describe this that the limit of (x^2-x+2) is 4 as x approaches2 from either side, and we write:



$$\lim_{x\to 2}(x^2-x+2)=4$$

Observe that in our investigation of $\lim_{x\to 2}(x^2-x+2)$ we are only concerned with the value of $f(x)$ near x=2 and not the value of $f(x)$ at x=2.

DEFINITION:

If the value of $f(x)$ can be made as close as we like to L by taking the value of x sufficiently close to a (but not equal a), then we write:

$$\lim_{x\to a} f(x)=L$$

Which is read "the limit of $f(x)$ as x approaches a is L".

Properties of limits:

1. If $f(x) = k,$ then $\lim_{x\to a} f(x) = k$ where a and k are real numbers.

2. If $\lim_{x\to a} f_1(x) = L_1$ and $\lim_{x\to a} f_2(x) = L_2$, then:

(*a*) Sum rule: $\lim_{x\to a} [f_1(x) + f_2(x)] = L_1 + L_2$

(*b*) Difference rule: $\lim_{x\to a} [f_1(x) - f_2(x)] = L_1 - L_2$

(*c*) Product rule: $\lim_{x\to a} [f_1(x).f_2(x)] = L_1.L_2$

(*d*) Constant multiple rule: $\lim_{x\to a} k.f_1(x) = k.L_1$ (where k is constant)

(*e*) Quotient rule: $\lim_{x\to a} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$; $L_2 \neq 0$

(*f*) Power rule: $\lim_{x\to a} [f_1(x)]^{r/s} = L_1^{r/s}$ (if s is even number $L_1 > 0$)

3. Polynomial $\lim_{x\to a} (c_0 + c_1x + c_2x^2 .....c_nx^n) = c_0 + c_1a + c_2a^2 .....c_na^n$

4. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

5. Sandwich theorem:

If $g(x) \le f(x) \le h(x)$ are three functions such that:

$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} f(x) = L$.

***Note*:**

1. For sake of convenience in dealing with indeterminate forms, we define the following arithmetic operations with real numbers, $+\infty$ and $-\infty$. Let c be a real number and $c > 0$. Then we define:

$+\infty + \infty = +\infty$, $-\infty - \infty = -\infty$, $c(+\infty) = +\infty$, $c(-\infty) = -\infty$, $(-c)(+\infty) = -\infty$, $(-c)(-\infty) = +\infty$, $\frac{c}{\infty} = 0$, $\frac{-c}{\infty} = 0$, $\frac{c}{-\infty} = 0$, $\frac{-c}{-\infty} = 0$, $(+\infty)^c = \infty$, $(+\infty)^{-c} = 0$, $(+\infty)(+\infty) = +\infty$, $(+\infty)(-\infty) = -\infty$, $(-\infty)(-\infty) = +\infty$

2. The following operations are indeterminate quantities:

$$\left(\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 0*\infty\right)$$

Examples: Find the limits of the following:

1. $\lim_{x\to 2} x^2 - 4x = 2^2 - 4*2 = 4 - 8 = -4$

2. $\lim_{x\to 1} x^3 + 2x^2 - 3x + 4 = 1^3 + 2*1^2 - 3*1 + 4 = 4$

3. $\lim_{x\to 1} \frac{(3x-1)^2}{(x+1)^3} = \frac{(3*1-1)^2}{(1+1)^3} = \frac{2^2}{2^3} = \frac{1}{2}$

4. $\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6} = \frac{2^2-4}{2^2-5*2+6} = \frac{0}{0}$ (Indeterminate quantities)

   So $\lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x\to 2} \frac{(x+2)}{(x-3)} = \frac{2+2}{2-3} = \frac{4}{-1} = -4$

5. $\lim_{x\to 2} \frac{x-2}{\sqrt{x^2-4}} = \frac{2-2}{\sqrt{2^2-4}} = \frac{0}{0}$ (Indeterminate quantities)

   $= \lim_{x\to 2} \frac{\sqrt{x-2}\sqrt{x-2}}{\sqrt{(x-2)(x+2)}} = \lim_{x\to 2} \frac{\sqrt{x-2}\sqrt{x-2}}{\sqrt{x-2}\sqrt{x+2}} = \lim_{x\to 2} \frac{\sqrt{x-2}}{\sqrt{x+2}} = \frac{\sqrt{2-2}}{\sqrt{2+2}} = \frac{0}{\sqrt{4}} = 0$

6. $\lim_{x\to 2} \frac{\sqrt{x-2}}{x^2-4} = \frac{\sqrt{2-2}}{2^2-4} = \frac{0}{0}$ (Indeterminate quantities)

   $= \lim_{x\to 2} \frac{\sqrt{x-2}}{(x-2)(x+2)} = \lim_{x\to 2} \frac{\sqrt{x-2}}{\sqrt{x-2}\sqrt{x-2}(x+2)}$

   $= \lim_{x\to 2} \frac{1}{\sqrt{x-2}(x+2)} = \frac{1}{\sqrt{2-2}(2+2)} = \frac{1}{0*4} = \frac{1}{0} = \infty$ So the limit does not exist

7. $\lim_{x\to1}\frac{x-1}{\sqrt{x^2+3}-2}=\frac{1-1}{\sqrt{1^2+3}-2}=\frac{0}{0}$ (Indeterminate quantities)

$=\lim_{x\to1}\frac{x-1}{\sqrt{x^2+3}-2}*\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2}$ (Multiplying both the numerator and denominator by conjugate factor)

$=\lim_{x\to1}\frac{(x-1)(\sqrt{x^2+3}+2)}{x^2+3-4}=\lim_{x\to1}\frac{(x-1)(\sqrt{x^2+3}+2)}{x^2-1}=\lim_{x\to1}\frac{(x-1)(\sqrt{x^2+3}+2)}{(x-1)(x+1)}$

$=\lim_{x\to1}\frac{(\sqrt{x^2+3}+2)}{(x+1)}=\frac{\sqrt{1^2+3}+2}{x+1}=\frac{4}{2}=2$

8. $=\lim_{x\to0}\frac{\sin 3x}{x}=\lim_{x\to0}\frac{\sin 3x}{x}*\frac{3}{3}=3*\lim_{x\to0}\frac{\sin 3x}{3x}=3*1=3$

Remember that $3x\to0$ when $x\to0$

9. $=\lim_{x\to0}\frac{\tan x}{x}=\lim_{x\to0}\frac{\sin x/\cos x}{x}=\lim_{x\to0}\frac{\sin x}{x}*\lim_{x\to0}\frac{1}{\cos x}=1*\frac{1}{1}=1$

10. $=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\frac{\pi}{2}-x}$ let $z=\frac{\pi}{2}-x$ so as $x\to\frac{\pi}{2}\Rightarrow z\to0$

$\therefore\lim_{z\to0}\frac{\cos(\frac{\pi}{2}-z)}{z}=\lim_{z\to0}\frac{\sin z}{z}=1$

11. $=\lim_{x\to0}\frac{1-\cos x}{x}=\lim_{x\to0}\frac{1-\cos x}{x}*\frac{1+\cos x}{1+\cos x}=\lim_{x\to0}\frac{1-\cos^2 x}{x(1+\cos x)}$

$=\lim_{x\to0}\frac{\sin^2 x}{x(1+\cos x)}=\lim_{x\to0}\frac{\sin x}{x}*\lim_{x\to0}\frac{\sin x}{(1+\cos x)}=1*\frac{0}{1+1}=0$

Homework: Find the limits of the following:

1. $\lim_{x\to\frac{1}{2}}\frac{2x^2+5x-3}{6x^2-7x+2}$
2. $\lim_{x\to2}\frac{x-2}{x^3-8}$
3. $\lim_{x\to16}\frac{x-16}{\sqrt{x}-4}$
4. $\lim_{x\to\frac{1}{2}}\frac{4x^2-6x+3}{16x^3+8x-7}$
5. $\lim_{x\to-3}\frac{x+3}{1/x+1/3}$
6. $\lim_{x\to2}\frac{x^2-x-2}{(x-2)^2}$

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7. $\lim_{x\to -2}\frac{x^3+8}{x^4-16}$

8. $\lim_{x\to 1}\frac{x^2}{x-1}-\frac{1}{x-1}$

9. $\lim_{h\to 0}\frac{4-\sqrt{16+h}}{h}$

10. $\lim_{h\to 0}(\frac{1}{h})(\frac{1}{\sqrt{1+h}}-1)$

11. $\lim_{x\to 0}\frac{x}{\sin x}$

12. $\lim_{x\to 0}\frac{x+\sin x}{x}$

13. $\lim_{x\to 0}\frac{\sin x}{2x^2-x}$

14. $\lim_{x\to 3}(x-3)\csc\pi x$

15. $\lim_{x\to 0}\frac{6x-\sin 2x}{2x+3\sin 4x}$

16. $\lim_{x\to 0}\frac{\sqrt[3]{8+x}-2}{x}$ (Hint: assume $\sqrt[3]{8+x}=z$)

17. $\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$