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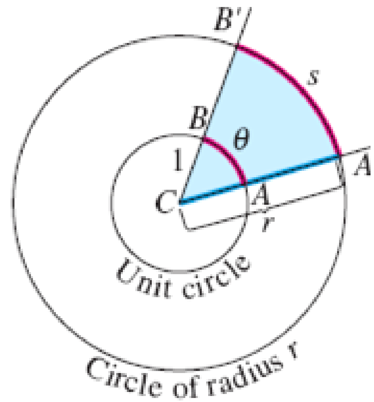
Trigonometric Functions

We measure angles in degrees, but in calculus it is usually best to use radians.

1. **Degree measure:** One degree (1^o) is the measure of an angle generated by 1/360 of revolution.
2. **Radian measure:** The radian measure of the angle *ACB* at the center of the unit circle (*circle with radius equals one unit*) equals to the length of the arc that the angle cuts from the unit circle.



If angle *ACB* cuts an arc *A`B`* from a second circle centered at C, then circular sector *A`CB`* will be similar to circular sector *ACB*. In particular,

$$\frac{Length\ of\ \ arc\ A`B`}{Radius\ of\ \ second\ circle} = \frac{Length\ of\ \ arc\ AB}{Radius\ of\ \ first\ circle}$$

In notations

$$\frac{s}{r} = \frac{\theta}{1} = \theta \qquad \Rightarrow \qquad \theta = \frac{s}{r} \qquad \text{or} \qquad s = r\theta$$

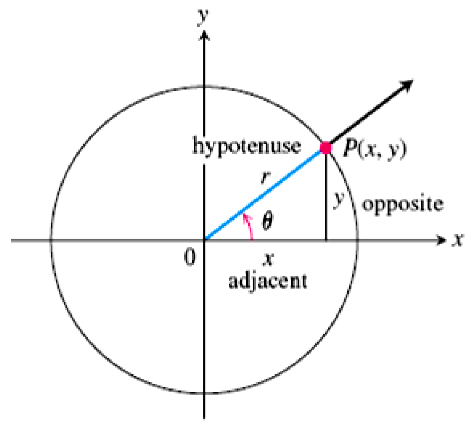
To find the relation between degree measure and radian measure, you know that one circle equals 360^o in degree and 2π in radians so:

2π radians= 360^o $\Rightarrow$ **π radians= 180^o**

$$\therefore \quad 1^o = \frac{\pi}{180} rad \cong 0.01745 rad$$

and $1 rad = \left(\frac{180}{\pi}\right)^o \cong 57^o 17' 44.8''$

The six basic trigonometric functions:



sine: $\sin\theta = \frac{y}{r}$,   cosecant: $\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$, cosine: $\cos\theta = \dfrac{x}{r}$, secant: $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{r}{x}$

tangent: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$, cotangent: $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta} = \dfrac{x}{y}$

from Pythagoras theorem:

$$x^2 + y^2 = r^2 \quad \Rightarrow \quad \frac{x^2+y^2}{r^2} = 1 \quad \Rightarrow \quad \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta = 1$ (true for all values of θ) ...(1)

When we divided eq.(1) by $\cos^2\theta$ yields:

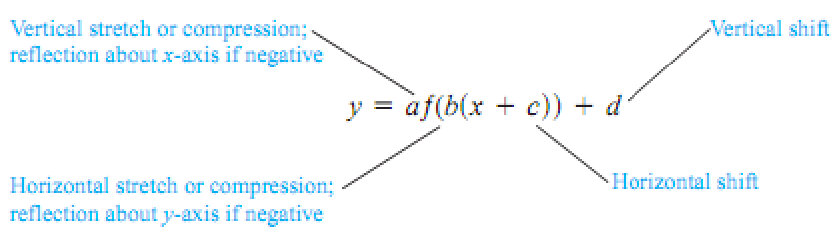
$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \Rightarrow \quad 1 + \tan^2\theta = \sec^2\theta$$

And when we divided eq.(1) by $\sin^2\theta$ yields:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \quad \Rightarrow \quad \cot^2\theta + 1 = \csc^2\theta$$

The rules for shifting, stretching, compressing, and reflecting:

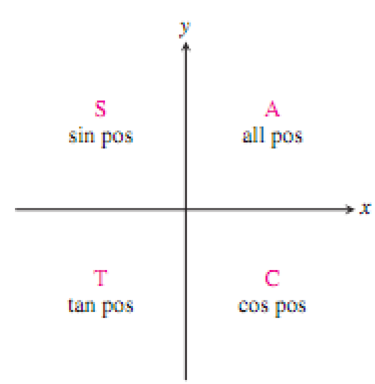
The rules for shifting, stretching, compressing, and reflecting the graph of a function apply to the trigonometric functions. The following diagram will remind you of the controlling parameters.



The CAST rule:

It is useful for remembering when the basic trigonometric functions are positive or negative.

The CAST rule, remembered by the statement "All Students Take Calculus," tells which trigonometric functions are positive in each quadrant.



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Identities:

- ***Periodicity*:** A function is periodic with period p if

  $f(x+p)= f(x)$ for every value of x.

  $\cos(\theta \pm 2\pi) = \cos\theta$

  $\sin(\theta \pm 2\pi) = \sin\theta$

  $\tan(\theta \pm 2\pi) = \tan\theta$

  $\cot(\theta \pm 2\pi) = \cot\theta$

  $\sec(\theta \pm 2\pi) = \sec\theta$

  $\csc(\theta \pm 2\pi) = \csc\theta$

- ***Symmetry*:**

Even functions	Odd functions
$\cos(-x)=\cos x$	$\sin(-x)=-\sin x$
$\sec(-x)=\sec x$	$\tan(-x)=-\tan x$
	$\cot(-x)=-\cot x$
	$\csc(-x)=-\csc x$

- ***Shift formulas:***

  $\sin(x + \pi/2) = \cos(x)$; $\cos(x + \pi/2) = -\sin(x)$

  $\sin(x - \pi/2) = -\cos(x)$; $\cos(x - \pi/2) = \sin(x)$

- ***Addition formulas:***

  $\cos(A+B)=\cos A \cos B - \sin A \sin B$

  $\sin(A+B)=\sin A \cos B + \cos A \sin B$

- ***Double angle formulas:***

  $\cos 2\theta=\cos^2\theta - \sin^2\theta$

  $\sin 2\theta= 2.\sin\theta.\cos\theta$

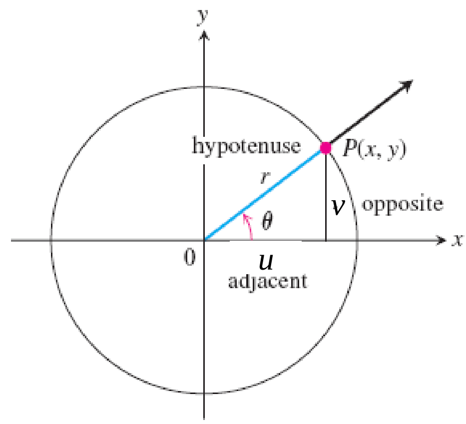
- ***Half angle formulas*:**

  $$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Graph of trigonometric functions:

1. $y = \sin x$



- Domain and Range of the function

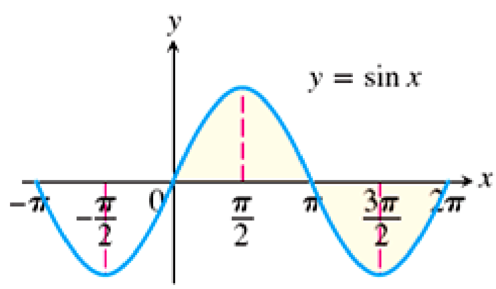
D_f=(-∞,∞)

From Figure nearby, we conclude that:

$-r \le v \le r$ (divide the inequality by r)

$$-1 \le \frac{v}{r} \le 1 \quad \Rightarrow \quad -1 \le \sin\theta \le 1$$

R_f=[-1,1]



- Symmetry:

$f(-x) = \sin(-x) = -\sin x = -f(x)$

$\neq f(x)$

So it is an odd function (it is symmetric about the origin).

- Additional points:

x	0	π/2	π	3π/2	2π
y	0	1	0	-1	0

2. $y = \cos x$

- Domain and Range of the function

D_f=(-∞,∞)

From Figure, we conclude that:

$-r \le u \le r$ (divide the inequality by r)

$$-1 \le \frac{u}{r} \le 1 \quad \Rightarrow \quad -1 \le \cos\theta \le 1$$

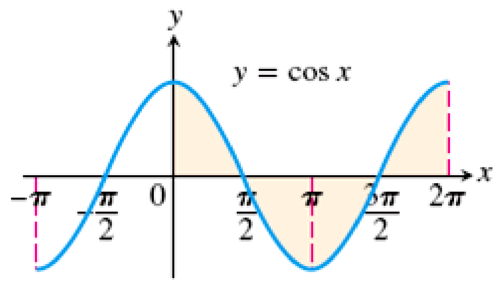
R_f=[-1,1]

- Symmetry:

$f(-x) = \cos(-x) = \cos x = f(x)$

$\neq - f(x)$

So it is an even function (it is symmetric about the y-axis).

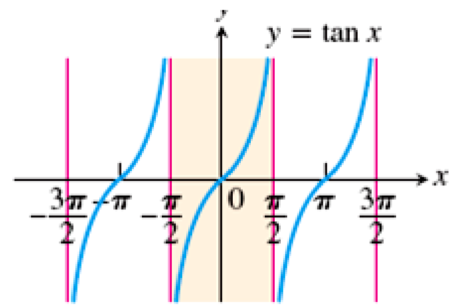


Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$

- Additional points:

x	0	π/2	π	3π/2	2π
y	1	0	-1	0	1

3. $y = \tan x = \dfrac{\sin x}{\cos x}$



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$

- Domain and Range of the function

$\cos x \neq 0 \quad \Rightarrow \quad x \neq n\frac{\pi}{2}$; $n=\pm1, \pm3, \pm5$

$D_f = R\backslash\{ x = n\frac{\pi}{2}$; $n=\pm1, \pm3, \pm5\}$

$R_f=(-\infty,\infty)$

- Symmetry:

$f(-x) = \tan(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin x}{\cos x} = -\tan x \neq f(x)$

$= - f(x)$

So it is an odd function (it is symmetric about the origin).

- Asymptotes:

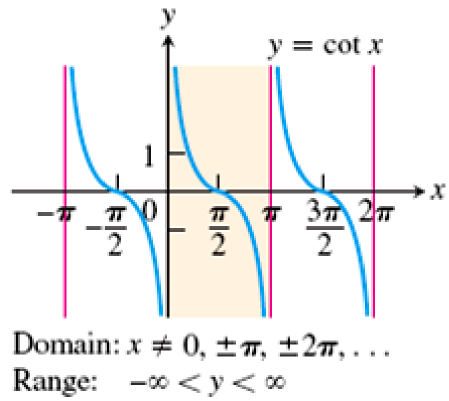
To find vertical asymptote put the denominator equal to zero.

$\cos x = 0 \quad \Rightarrow \quad x = n\frac{\pi}{2}$; $n=\pm1, \pm3, \pm5$

- Additional points:

x	0	π/2	π	3π/2	2π
y	0	±∞	0	±∞	0

4. $y = \cot x = \dfrac{\cos x}{\sin x}$



- Domain and Range of the function

$\sin x \neq 0 \quad \Rightarrow \quad x = n\pi$; n=0, ±1, ±2, ±3

$D_f = R \backslash \{x = n\pi;\ n = 0, \pm1, \pm2, \pm3\}$

$R_f = (-\infty, \infty)$

- Symmetry:

$f(-x) = \cot(-x) = \dfrac{\cos(-x)}{\sin(-x)} = \dfrac{\cos x}{-\sin x} = -\cot x \neq f(x)$

$= -f(x)$

So it is an odd function (it is symmetric about the origin).

- Asymptotes:

To find vertical asymptote put the denominator equal to zero.

$\sin x = 0 \quad \Rightarrow \quad x \neq n\pi$; n=0, ±1, ±2, ±3

- Additional points:

x	0	π/2	π	3π/2	2π
y	±∞	0	±∞	0	±∞

5. $y = \sec x = \dfrac{1}{\cos x}$

- Domain and Range of the function

$\cos x \neq 0 \quad \Rightarrow \quad x \neq n\dfrac{\pi}{2}$; n=±1, ±3, ±5

$D_f = R \backslash \{ x = n\dfrac{\pi}{2};\ n = \pm1, \pm3, \pm5\}$

From Figure, we conclude that:

$-r \leq u \leq r$ (divide the inequality by r)

$$-1 \leq \frac{u}{r} \leq 1 \quad \Rightarrow \quad -1 \leq \cos\theta \leq 1 \quad \Rightarrow \quad |\cos\theta| \leq 1 \quad \Rightarrow \quad \left|\frac{1}{\cos\theta}\right| \geq 1$$

$\Rightarrow |\sec\theta| \geq 1 \Rightarrow \quad \sec\theta \geq 1 \quad$ or $\quad \sec\theta \leq -1$

$R_f = R \backslash (-1,1)$

- Symmetry:

$f(-x) = \sec(-x) = \dfrac{1}{\cos(-x)} = \dfrac{1}{\cos x} = \sec x = f(x)$

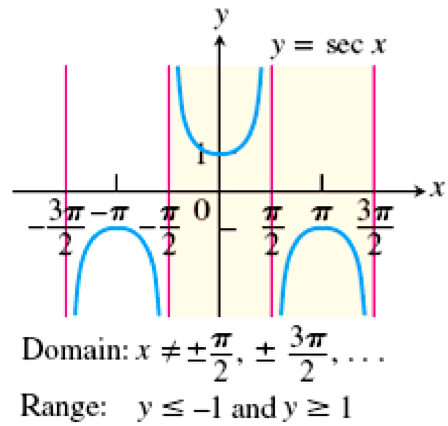
$\neq - f(x)$

So it is an even function (it is symmetric about the y-axis).

- Asymptotes:

To find vertical asymptote put the denominator equal to zero.

$\cos x = 0 \quad \Rightarrow \quad x = n\dfrac{\pi}{2}$; $n = \pm1, \pm3, \pm5$



- Additional points:

x	0	π/2	π	3π/2	2π
y	1	±∞	-1	±∞	1

6. $y = \csc x = \dfrac{1}{\sin x}$

- Domain and Range of the function

$\sin x \neq 0 \quad \Rightarrow \quad x \neq n\pi$, $n = 0, \pm1, \pm2, \pm3$

$D_f = R \backslash \{x = n\pi;\ n = 0, \pm1, \pm2, \pm3\}$

From Figure, we conclude that:

$-r \leq v \leq r$ (divide the inequality by r)

$-1 \leq \dfrac{v}{r} \leq 1 \quad \Rightarrow \quad -1 \leq \sin\theta \leq 1 \quad \Rightarrow \quad |\sin\theta| \leq 1 \quad \Rightarrow \quad \left|\dfrac{1}{\sin\theta}\right| \geq 1$

$\Rightarrow |\csc\theta| \geq 1 \Rightarrow \quad \csc\theta \geq 1 \quad$ or $\quad \csc\theta \leq -1$

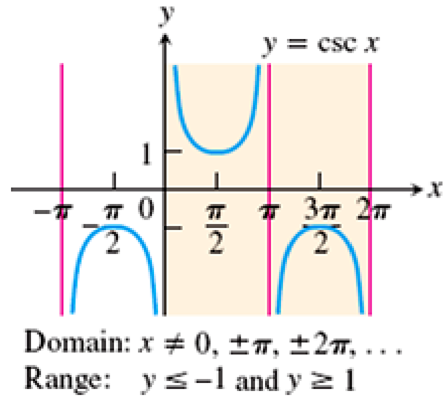
$R_f = R \backslash (-1,1)$

- Symmetry:

$f(-x) = \csc(-x) = \dfrac{1}{\sin(-x)} = \dfrac{1}{-\sin x} = -\csc x \neq f(x)$

$= -f(x)$

So it is an odd function (it is symmetric about the origin).



- Asymptotes:

To find vertical asymptote put the denominator equal to zero.

$\sin x = 0 \quad \Rightarrow \quad x = n\pi\,;\ n = 0, \pm1, \pm2, \pm3$

- Additional points:

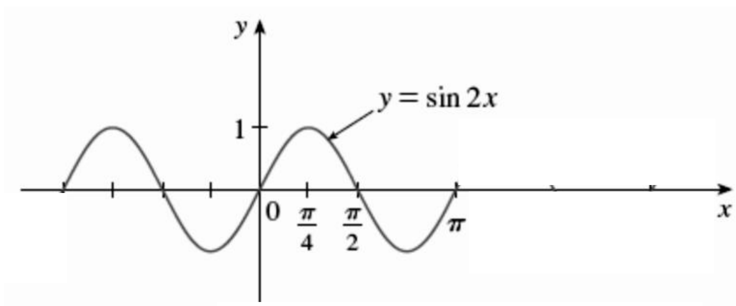
x	0	π/2	π	3π/2	2π
y	±∞	1	±∞	-1	±∞

Example: Sketch the graphs of the following functions.

1. $y = \sin 2x$

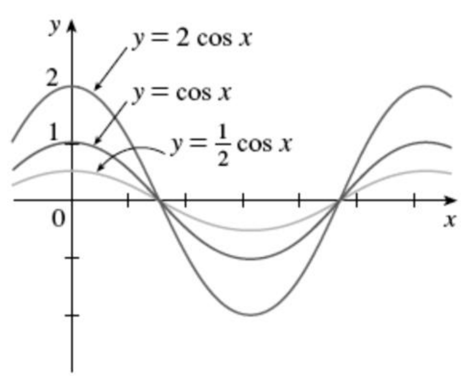
Sol.: We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by shrinking it two units horizontally.

$D_g = (-\infty, \infty)$; $R_g = [-1, 1]$



2. $y = 2\cos x$

Sol.: In order to get the graph of $y = 2\cos x$ we multiply the y-coordinate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

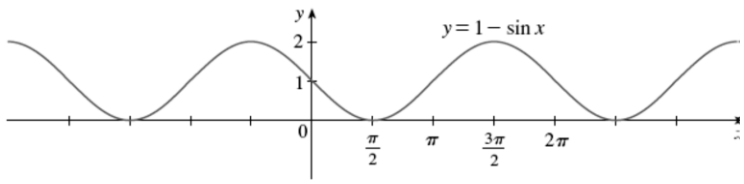


$D_g = (-\infty, \infty)$; $R_g = \{y : -1 \le \frac{y}{2} \le 1\}$

$= \{y : -2 \le y \le 2\} = [-2, 2]$

3. $y = 1 - \sin x$

Sol.: To obtain the graph of y=1-sin$x$, we again start with y=sin$x$. We reflect across the x-axis to get the graph of and then we shift 1 unit upward to get y=1-sin$x$.



D_g=(-∞,∞); R_g={y: -1 ≤ y-1 ≤1}

={y: 0 ≤ y ≤2} = [0, 2]

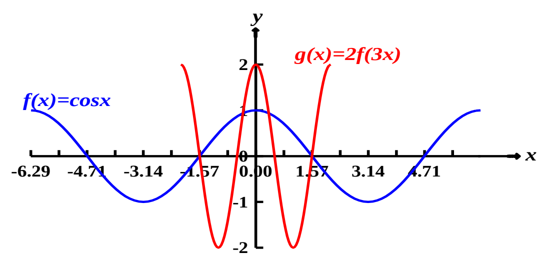
4. y= 2cos3x, $-2\pi \le 3x \le 2\pi$

Sol.: We obtain the graph of $y=g(x)=2\cos 3x$ from that of $y=f(x)=\cos x$ by shrinking it three units horizontally and stretching it two units vertically.

$D_g=\{x: -2\pi \le 3x \le 2\pi\} = \{x: -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}\}$

$=[-\frac{2\pi}{3}, \frac{2\pi}{3}]$

$R_g=\{y: -1 \le \frac{y}{2} \le 1\}=\{y: -2 \le y \le 2\} = [-2, 2]$



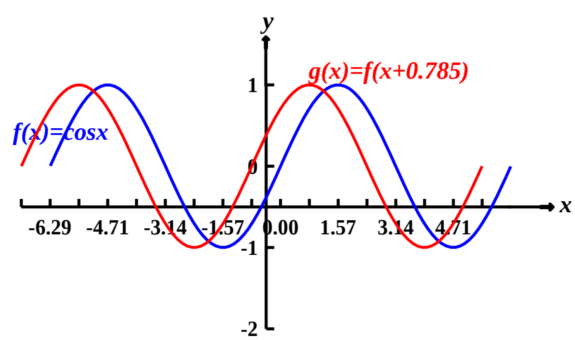
5. $y=\sin(x+\frac{\pi}{4})$, $-2\pi \le x \le 2\pi$

Sol.: We obtain the graph of $y=g(x) = \sin(x+\frac{\pi}{4})$ from that of $y=f(x)=\sin x$ by shifting it $\frac{\pi}{4}$ units left.

$D_g=\{x: -2\pi \le x+\frac{\pi}{4} \le 2\pi\} = \{x: -\frac{9\pi}{4} \le x \le \frac{7\pi}{4}\}$

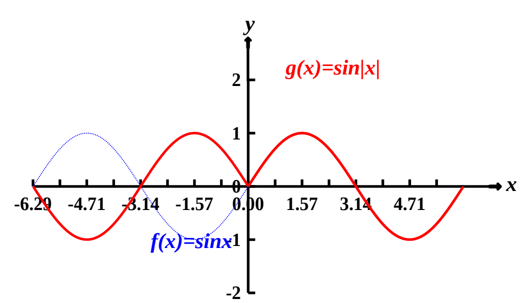
$=[-\frac{9\pi}{4}, \frac{7\pi}{4}]$

While the range will not be changed, R_g=[-1,1]



6. $y = \sin|x|$

Sol.: $y = \sin|x| = \begin{cases} \sin(x) & if \quad (x) \ge 0 \\ \sin(-x) = -\sin x & if \quad (x) < 0 \end{cases}$



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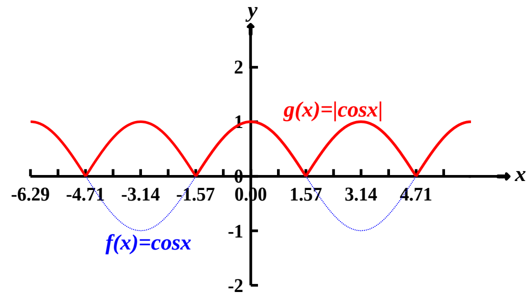
$D_g = (-\infty,\infty)$; $R_g=[-1,1]$

7. $y= |\cos x|$

Sol.: $y = |\cos x| = \begin{cases} (\cos x) & if \quad (\cos x) \geq 0 \\ -(\cos x) & if \quad (\cos x) < 0 \end{cases}$

$D_g=(-\infty,\infty)$; $R_g=[0,1]$



Homework: Graph the following functions and show domains and ranges of them.

1. $y = \sin 2x$
2. $y = 2\tan x$
3. $y = \cot 3x$
4. $y = \frac{1+\cos 2x}{2}$
5. $y = |x^2 + 1|$
6. $y = \sin \frac{x}{3}$
7. $y = 2\cos \frac{x}{3}$
8. $y = |x^2 - 4|$
9. $y = 2\sec x$
10. $y = \sqrt{(2x)^2}$
11. $y = \frac{x+|x|}{2}$
12. $y = \frac{\cos x + |\cos x|}{2}$
13. $y = |2x+1|$
14. $y = \frac{3x+|x|}{x}$
15. $y = \frac{|x|}{x^2}$
16. $y = \frac{|\sin x|}{\cos x}$
17. $y = \frac{x^3+|x|}{x}$