syllabus

Composition of Functions:

Definition: If f and g are functions, the composite $f \circ g$ "f composed with g" or $g \circ f$ "g composed with f" are defined by:

$(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$ respectively.

Examples 1: Find the formula for $f(g(x))$ and $g(f(x))$ if $g(x) = x^2$ and $f(x) = x-7$, then find the value of $f(g(2))$ and $g(f(2))$

Sol.: a: For $f(g(x))=?$, $f(x) = x-7$

$f(g(x)) = (g(x))-7 = x^2-7 \quad \therefore \quad f(g(2)) = 2^2-7 = 4-7 = -3$

b: For $g(f(x))=?$, $g(x) = x^2$

$g(f(x)) = (f(x))^2 = (x-7)^2 \quad \therefore \quad g(f(2)) = (2-7)^2 = (-5)^2 = 25$

Examples 2: If $f(x) = x^2+1$ and $g(x) = \sqrt{x}$, find

a: $(f \circ g)(x)$ and $(g \circ f)(x)$?

b: the domains of $(f \circ g)(x)$ and $(g \circ f)(x)$?

Sol.: $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1 \quad \Rightarrow \quad D_{f \circ g} = D_g = \{x : x \geq 0\}$

$(g \circ f)(x) = g(f(x)) = g(x^2+1) = \sqrt{x^2+1} \quad \Rightarrow \quad D_{g \circ f} = D_f = R$

Examples 3: Finding formulas for composites:

If $f(x) = \sqrt{x}$ *and* $g(x) = x+1$, *find*

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$

Sol.:

Composite	Domain
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	[-1, ∞)
(b) $(g \circ f)(x) = g(f(x)) = f(x)+1 = \sqrt{x}+1$	[0, ∞)
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	[0, ∞)
(d) $(g \circ g)(x) = g(g(x)) = g(x)+1 = (x+1)+1 = x+2$	(-∞, ∞)

Example 4: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each function and its domain

(a) $f_o g$ (b) $g_o f$ c) $f_o f$ (d) $g_o g$

Sol.:

(a) $(f_o g)(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$

The domain of $f_o g$ $=\{x: 2-x \geq 0\}=\{x : x \leq 2\}=(-\infty,2]$.

(b) $(g_o f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$

For $\sqrt{x}$ to be defined we must have $x \geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have $2-\sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus, we have $0 \leq x \leq 4$, so the domain of $g_o f$ is the closed interval $[0,4]$.

(c) $(f_o f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$

The domain of $f_o f$ is $[0,\infty)$

(d) $(g_o g)(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$

This expression is defined when $2 - x \geq 0$, that is $x \leq 2$, and $2-\sqrt{2-x} \geq 0$. This latter inequality is equivalent to $\sqrt{2-x} \leq 2$, or $2 - x \leq 4$, that is, $x \geq -2$. Thus, $-2 \leq x \leq 2$, so the domain of $g_o g$ is the closed interval $[-2,2]$.

Homework: **Find $(f_o g)$, $(g_o f)$, $(f_o f)$ and $(g_o g)$ of the following functions:**

1. $f(x) = x^3$; $g(x) = x^2 + 3$.
2. $f(x) = 3x^2 + 2$; $g(x) = \frac{1}{3x^2 + 2}$.
3. $f(x) = \sqrt{2x+1}$; $g(x) = x^2 + 3$.
4. $f(x) = 7$; $g(x) = 4$.
5. $f(x) = x^3$; $g(x) = \sin x - 3$.
6. $f(x) = \frac{1}{1+x}$; $g(x) = \sqrt[3]{x}$.
7. $f(x) = 3x^2 + x$; $g(x) = 2x - 1$.

8. If $f(x) = x + 5$ and $g(x) = x^2 - 3$; Find

(a) $f(g(0))$ (b) $g(f(0))$ (c) $f(f(-5))$ (d) $g(g(2))$

Graph of Functions (Graph of Curves):

To graph the curve of a function, we can follow the following steps:

1. Find the domain and range of the function.
2. Check the symmetry of the function
3. Find (if any found) points of intersection with x-axis and y-axis.
4. Choose some another points on the curve.
5. Draw s smooth line through the above points.

Symmetry Tests for Graphs:

If $f(x,y) = 0$ is any function then:

1. Symmetry about ***x*-axis**: If $f(x,-y)= f(x,y)$

2. Symmetry about ***y*-axis**: If $f(-x,y)=f(x,y)$ It is called an *even* function.

3. Symmetry about **the origin**: If $f(-x,-y)=f(x,y)$ It is called an *odd* function

Examples 1: Check the symmetry of the graph of the following curves:

1. $y = f(x)=x^2$

sol.: $f(x,y) = x^2- y =0$

$\therefore$ (*i*) $f(x,-y)= x^2- (-y) =0 \quad \Rightarrow \quad . f(x,-y)= x^2+ y =0 \neq f(x,y)$ not o.k.

(*ii*) $f(-x,y)= (-x)^2- y =0 \quad \Rightarrow \quad . f(-x,y)= x^2- y =0= f(x,y)$ o.k.

(*iii*) $f(-x,-y)= (-x)^2-(-y)=0 \quad \Rightarrow \quad . f(-x,-y)= x^2+ y =0 \neq f(x,y)$ not o.k.

So the function has symmetry only about y-axis. It is called an even function.

2. $y= f(x)=x^3$

Sol. $f(x,y) = x^3- y =0$

$\therefore$ (*i*) $f(x,-y)= x^3- (-y) =0 \quad \Rightarrow \quad . f(x,-y)= x^3+ y =0 \neq f(x,y)$ not o.k.

(*ii*) $f(-x,y)= (-x)^3- y =0 \quad \Rightarrow \quad . f(-x,y)= -x^3- y =0 \neq f(x,y)$ not o.k.

(*iii*) $f(-x,-y)= (-x)^3-(-y)=0 \quad \Rightarrow \quad . f(-x,-y)= -x^3+ y =0$ (multiply by -1)

$\Rightarrow \quad f(-x,-y) = x^3 - y = 0 = f(x,y)$ o.k.

So the function has symmetry only about *the origin*. It is called an odd function.

3. $x^2 = y^2 + 4$

Sol. $f(x,y) = y^2 - x^2 + 4 = 0$

$\therefore$ (*i*) $f(x,-y) = (-y)^2 - x^2 + 4 = 0 \quad \Rightarrow \quad . f(x,-y) = y^2 - x^2 + 4 = 0 = f(x,y)$ o.k.

(*ii*) $f(-x,y) = y^2 - (-x)^2 + 4 = 0 \quad \Rightarrow \quad . f(-x,y) = y^2 - x^2 + 4 = 0 = f(x,y)$ o.k.

(*iii*) $f(-x,-y) = (-y)^2 - (-x)^2 + 4 = 0 \quad \Rightarrow \quad . f(-x,-y) = y^2 - x^2 + 4 = 0 = f(x,y)$ o.k.

So the function has symmetry about *x*-axis, *y*-axis and *the origin*.

DEFINITIONS **Even Function, Odd Function**

A function $y = f(x)$ is an

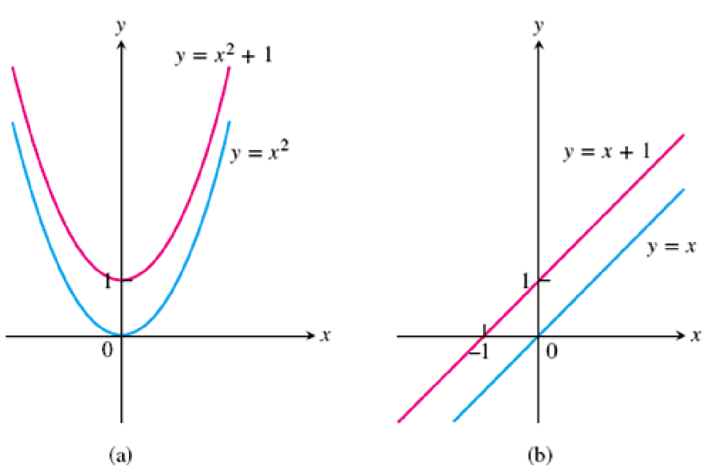
even function of *x* if $f(-x) = f(x)$ symmetry about y-axis

odd function of *x* if $f(-x) = -f(x)$ symmetry about origin

for every *x* in the function's domain.

Examples 2: Recognizing Even and Odd functions

- $f(x) = x^2$ Even function: $(-x)^2 = x^2$ for all *x*; symmetry about *y*-axis.
- $f(x) = x^2 + 1$ Even function: $(-x)^2 + 1 = x^2 + 1$ for all *x*; symmetry about *y*-axis.
- $f(x) = x$ Odd function: $(-x) = x$ for all *x*; symmetry about *the origin*.
- $f(x) = x + 1$ Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.
  Not even: $f(-x) = -x + 1$, but $f(x) = x + 1$. for all $x \neq 0$.



Example 3: Sketch the graph of the curve $y = f(x) = x^2 - 1$

Sol.: Step 1: Find D_f, R_f of the function?

$D_f = (-\infty, \infty)$;

To find R_f : we must convert the function from $y=f(x)$ into $x=f(y)$.

$y = x^2 - 1 \Rightarrow x^2 = y+1 \quad x = \mp\sqrt{y+1}$

So $y + 1 \geq 0 \Rightarrow y \geq -1 \Rightarrow R = [-1, \infty)$

Step 2: Find x and y intercept?

To find x-intercept put $y=0 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 = \pm 1$

So x-intercept are $(-1,0)$ and $(+1,0)$.

To find y-intercept put $x=0 \Rightarrow y = 0-1 \Rightarrow y = -1$

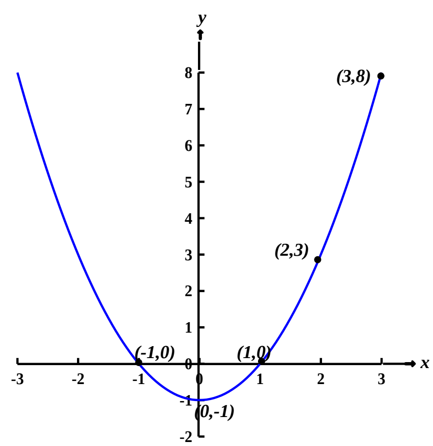
So y-intercept is $(0,-1)$.

Step 3: check the symmetry:

$f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$

$-f(x) = -(x^2 - 1) = -x^2 + 1 \neq f(x)$

So it is an even function (it is symmetric about y-axis).



Step 4: Choose some another point on the curve.

x	y
2	3
3	8

Step 5: Draw smooth line through the above points.

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Homework: Draw the following functions:

1. $y = f(x) = 3x^2 + 2$
2. $x^2 + y^2 = 1$
3. $y^2 = 4x - 1$
4. $x = y^3$
5. $y = [x]$; for $-3 \le x \le 3$
6. $y = x - [x]$; for $-2 \le x \le 2$
7. $y = \sqrt{4 - x}$

Shifting, Shrinking and Stretching:

Shift formulas: (for c > 0)

Vertical shifts

$y= f(x)+c$	or	$y-c= f(x)$	shifts the graph of f up by c units.
$y= f(x)-c$	or	$y+c= f(x)$	shifts the graph of f down by c units.

Horizontal shifts

$y= f(x+c)$	shifts the graph of f left by c units.
$y= f(x-c)$	shifts the graph of f right by c units.

Shrinking, Stretching and Reflecting Formulas:

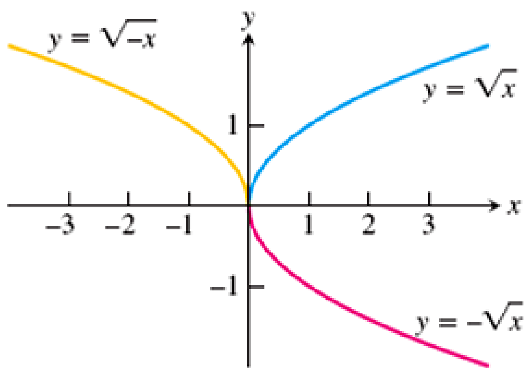
(for c > 1)

$y=c\,f(x)$	Stretches the graph of f c units along y-axis.
$y=\frac{1}{c}f(x)$	Shrinks the graph of f c units along y-axis.
$y= f(cx)$	Shrinks the graph of f c units along x-axis.
$y=f(\frac{x}{c})$	Stretches the graph of f c units along x-axis.

(for c = -1)

$y=- f(x)$	Reflects the graph of f across the x-axis.
$y= f(-x)$	Reflects the graph of f across the y-axis.

Example 1: The graph of $y=-\sqrt{x}$ is a reflection of $y=\sqrt{x}$ across the x-axis, and $y=\sqrt{-x}$ is a reflection across the y-axis.



Example 2: Shift the graph of the function

$f(x) = x^2$; if $D_f=\{x: -2 \le x \le 3\}$ and $R_g=\{y: 0 \le y \le 9\}$.

(a) one unit right. (b) two units left.

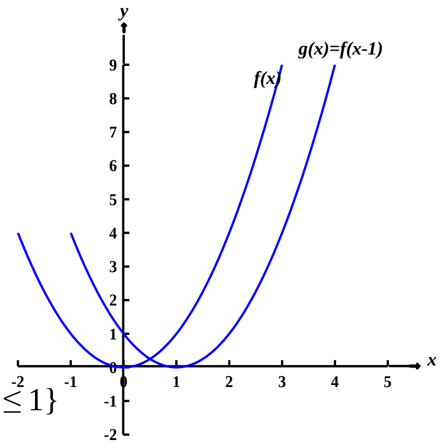
(c) one unit up. (d) two units down.

Sol.: (a) Shifting the function $f(x)$ one unit right:

$g(x) = f(x-1) = (x-1)^2$ and $D_g=\{x: -2 \le x-1 \le 3\}=\{x: -1 \le x \le 4\}$

Note: In case of horizontal shifts, the range of the function will not be changed.

x	$y=f(x)=x^2$	$x-1$	$y=g(x)=(x-1)^2$
-2	4	-	**-**
-1	1	-2	**4**
0	0	-1	**1**
1	1	0	**0**
2	4	1	**1**
3	9	2	**4**
4	**-**	**3**	**9**

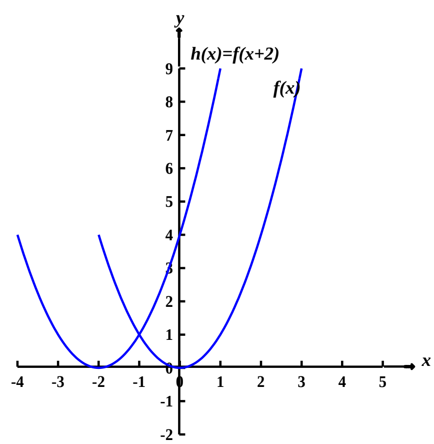


(b) Shifting the function $f(x)$ two units left:

$h(x) = f(x+2) = (x+2)^2$ and $D_h=\{x: -2 \le x+2 \le 3\}=\{x: -4 \le x \le 1\}$

Note: In case of horizontal shifts, the range of the function will not be changed.

X	$y=f(x)=x^2$	$x+2$	$y=h(x)=(x+2)^2$
-4	-	-2	**4**
-3	-	-1	**1**
-2	4	0	**0**
-1	1	1	**1**
0	0	2	**4**
1	1	3	**9**
2	4	-	**-**
3	**9**	**-**	

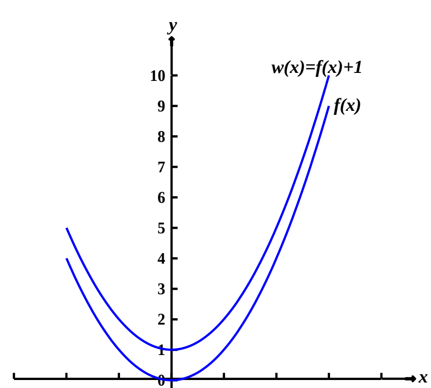


(c) Shifting the function $f(x)$ one unit up:

$w(x) = f(x)+1 = x^2 +1$ and $R_w=\{y: 0 \le y-1 \le 9\}=\{y: 1 \le y \le 10\}$

Note: In case of vertical shifts, the domain of the function will not be changed.

X	$y=f(x)=x^2$	$y=w(x)= x^2 +1$
-2	4	**5**
-1	1	**2**
0	0	**1**



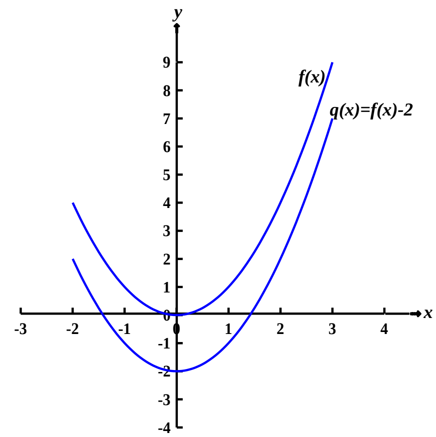
1	1	**2**
2	4	**5**
3	**9**	**10**

(d) Shifting the function $f(x)$ two units down:

$q(x) = f(x)-2 = x^2 -2$ and $R_q=\{y: 0 \le y+2 \le 9\}=\{y: -2 \le y \le 7\}$

Note: In case of vertical shifts, the domain of the function will not be changed.

X	$y=f(x)=x^2$	$y=q(x)= x^2 -2$
-2	4	**2**
-1	1	**-1**
0	0	**-2**
1	1	**-1**
2	4	**2**
3	**9**	**7**



Example 3: Sketch the graph of the curve $y= f(x) = |x|$

Sol.: **Step1**: Find D_f, R_f of the function?

$$y = f(x) = |x| = \begin{cases} x & if \quad x \ge 0 \\ -x & if \quad x < 0 \end{cases}$$

$\Rightarrow$ $D_f=(-\infty,\infty)$ and $R_f= [0,\infty)$;

Step2: Find x and y intercept?

To find x-intercept put $y=0 \Rightarrow x=0$

To find y-intercept put $x=0 \Rightarrow y=0$

So x- and y-intercept is (0,0).

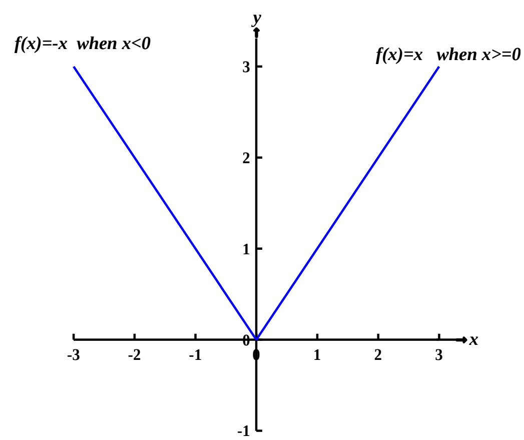
Step 3: check the symmetry:

$f(-x) = |-x|=|x|= f(x)$

$- f(x) =-|x| \neq f(x)$

So it is an even function (it is symmetric about y-axis).

Step 4: Choose some another point on the curve.

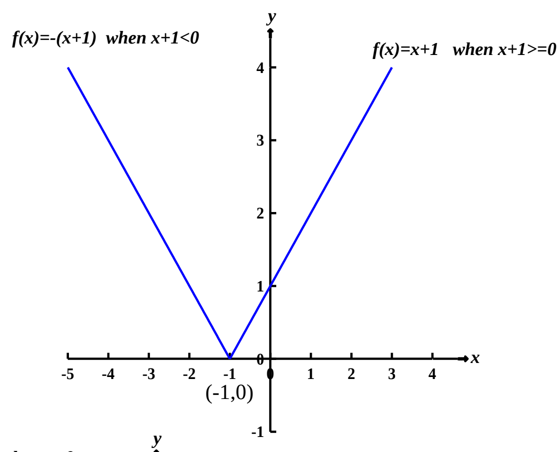


x	y
1	1
2	2

Step 5: Draw smooth line through the above points.

Example 4: Use graph of the function $y=|x|$ to sketch the graph of the following functions, then show their domains and range

(a) $y=|x+1|$



Sol.

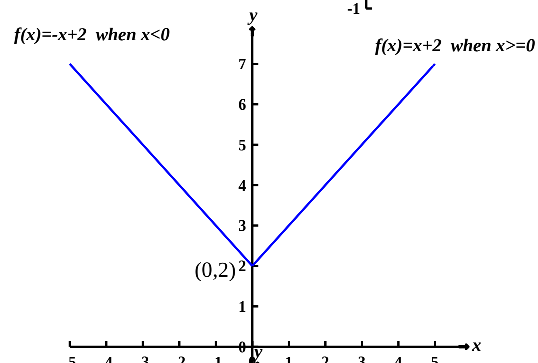
$$y=|x+1|=\begin{cases}(x+1) & if \quad (x+1)\geq 0\\ -(x+1) & if \quad (x+1)<0\end{cases}$$

$$=\begin{cases}(x+1) & if \quad x\geq -1\\ -x-1 & if \quad x<-1\end{cases}$$

Shifting the function $y=|x|$ one unit left.

$D_f=(-\infty,\infty)$ and $R_f=[0,\infty)$

(b) $y=|x|+2$

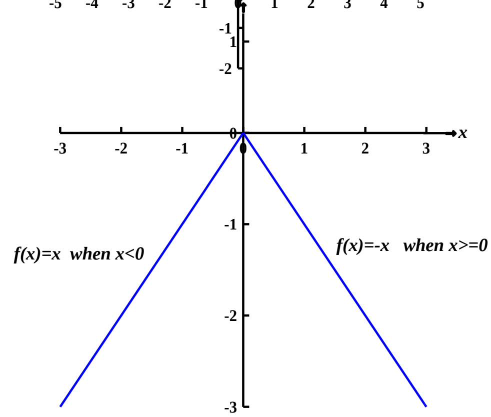


Sol. $y==|x|+2=\begin{cases}(x)+2 & if \quad (x)\geq 0\\ (-x)+2 & if \quad (x)<0\end{cases}$

Shifting the function $y=|x|$ two up.

$D_f=(-\infty,\infty)$ and $R_f=[2,\infty)$

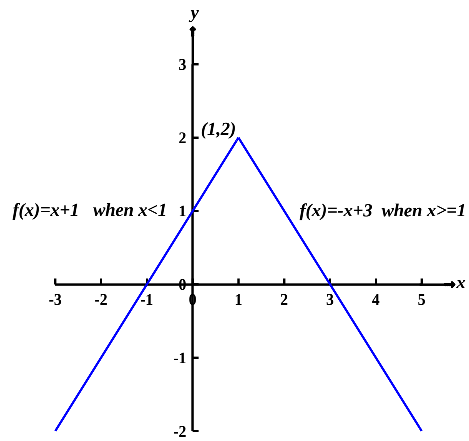
(c) $y=-|x|$



Sol. $y=f(x)=-|x|=\begin{cases}-(x)=-x & if \quad (x)\geq 0\\ -(-x)=x & if \quad (x)<0\end{cases}$

Reflecting the graph of the function $y=|x|$ across x-axis.

$D_f=(-\infty,\infty)$ and $R_f=(-\infty,0]$



(d) $y=2-|1-x|$

Sol. $y=2-|1-x|=-|1-x|+2|=-|x-1|+2$

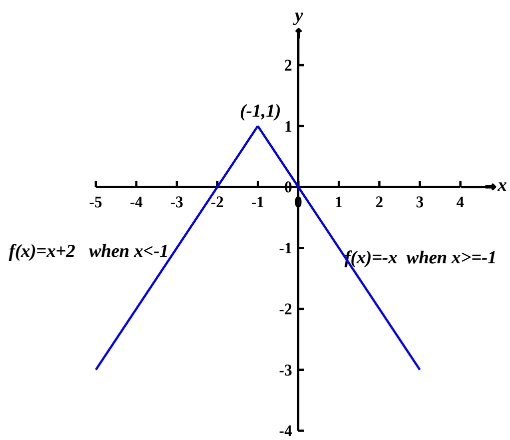
$$=\begin{cases} -(x-1)+2 & if \quad (x-1)\geq 0 \\ -(-(x-1))+2 & if \quad x-1<0 \end{cases}$$

$$=\begin{cases} -x+3 & if \quad x\geq 1 \\ x+1 & if \quad x<1 \end{cases}$$

Reflecting the graph of the function $y=|x|$ across x-axis, then shifting it one unit right and two units up.

$D_f=(-\infty,\infty)$ and $R_f=(-\infty,2]$

(e) $y=1-|x+1|$



Sol. $y=1-|x+1|=-|x+1|+1$

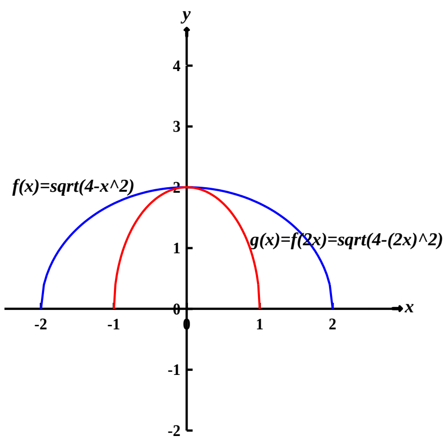
$$=\begin{cases} -(x+1)+1 & if \quad (x+1)\geq 0 \\ -(-(x+1))+1 & if \quad (x+1)<0 \end{cases}$$

$$=\begin{cases} -x & if \quad x\geq -1 \\ x+2 & if \quad x<-1 \end{cases}$$

Reflecting the graph of the function $y=|x|$ across x-axis, then shifting it one unit left and one unit up.

$D_f=(-\infty,\infty)$ and $R_f=(-\infty,1]$

Example 5: If $f(x)=\sqrt{4-x^2}$ which has $D_f=[-2,2]$ and $R_f=[0,2]$, shrink and stretch it horizontally by two units and then



sketch the original and resulting functions

Sol.: (a) shrinking:

$$g(x) = f(c.x) = \sqrt{4-(2x)^2} = \sqrt{4-4x} = 2\sqrt{1-x^2}$$

D_g={x: -2 ≤ 2x ≤ 2}={x: -1≤ x ≤ 1}

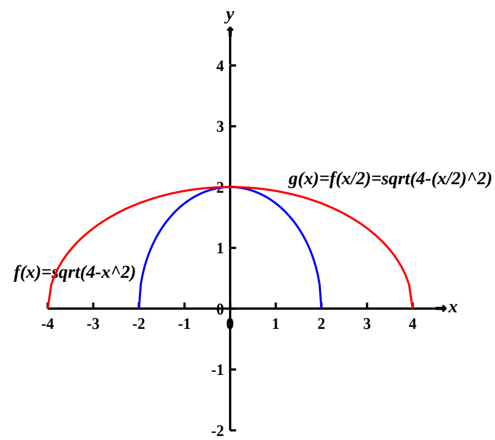
Note: In case of horizontal shrinks, the range of the function will not be changed.

(b) stretching:

$$g(x) = f(\frac{x}{c}) = \sqrt{4-(\frac{x}{2})^2} = \sqrt{4-\frac{x^2}{4}} = \sqrt{\frac{16-x^2}{4}} = \frac{1}{2}\sqrt{16-x^2}$$

D_g={x: -2 ≤ x/2 ≤ 2}={x: -4≤ x ≤ 4}

Note: In case of horizontal stretches, the range of the function will not be changed.



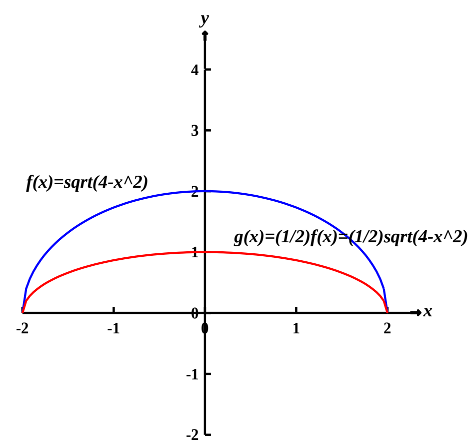
Example 6: Repeat the above example but here shrink and stretch the function vertically.

Sol.: (a) shrinking:

$$g(x) = \frac{1}{c} f(.x) = \frac{1}{2}\sqrt{4-x^2}$$

R_g={y: 0 ≤ 2y ≤ 2}={y: 0 ≤ y ≤ 1}

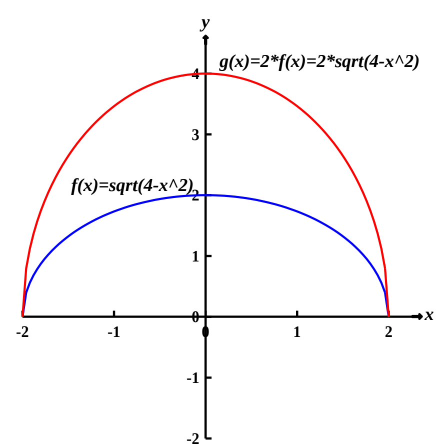
Note: In case of vertical shrinks, the domain of the function will not be changed.



(b) stretching:

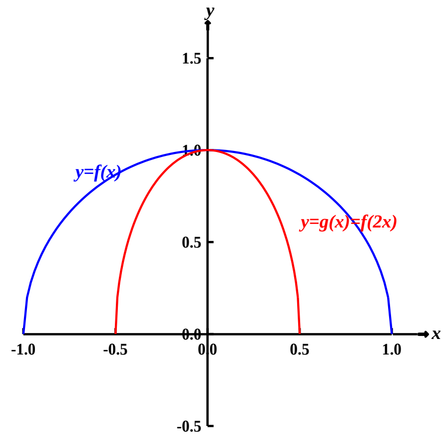
$$g(x) = cf(.x) = 2\sqrt{4-x^2}$$

R_g={y: 0 ≤ y/2 ≤ 2}={y: 0 ≤ y ≤ 4}



Note: In case of vertical stretches, the domain of the function will not be changed.

Example 7: Use the graph of the function $y = f(x) = \sqrt{1-x^2}$ to sketch the graph of the following functions:



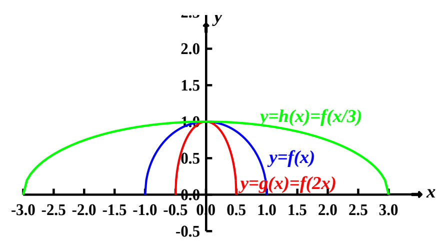
1. $y = g(x) = \sqrt{1-4x^2}$

Sol.: $y = \sqrt{1-4x^2} = \sqrt{1-(2x)^2}$

This function may be obtained by shrinking the function $f(x) = \sqrt{1-x^2}$ by two units horizontally ($g(x) = f(2x)$).

2. $y = h(x) = \sqrt{1 - \dfrac{x^2}{9}}$

Sol.: $y = \sqrt{1-\dfrac{x^2}{9}} = \sqrt{1-(\dfrac{x}{3})^2}$

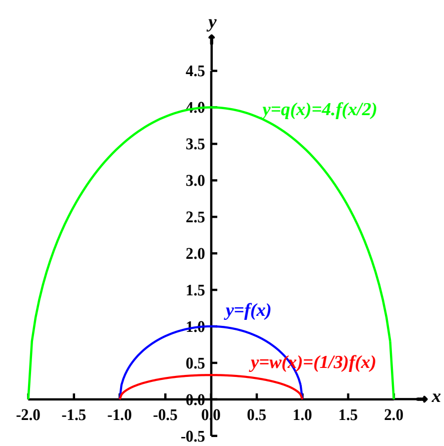


This function may be obtained by stretching the function $f(x) = \sqrt{1-x^2}$ by three units horizontally ($h(x) = f(\dfrac{x}{3})$).

3. $y = w(x) = \dfrac{1}{3}\sqrt{1-x^2}$

Sol.: $y = w(x) = \dfrac{1}{3}\sqrt{1-x^2}$

This function may be obtained by shrinking the function $f(x) = \sqrt{1-x^2}$ by three units vertically ($h(x) = \dfrac{1}{3} f(x)$).



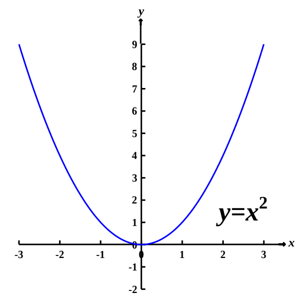
4. $y = q(x) = 4\sqrt{1 - \frac{x^2}{4}}$

Sol.: $y = q(x) = 4\sqrt{1 - \frac{x^2}{4}} = 4\sqrt{1 - (\frac{x}{2})^2}$

This function may be obtained by stretching the function $f(x) = \sqrt{1 - x^2}$ by two units horizontally and four units vertically ($q(x) = 4.f(\frac{x}{2})$).

Homework:

1. Sketch the graph of the following curves by shifting, reflecting, shrinking and stretching the graph of the given functions appropriately.



(a) The given function $y=x^2$

(i) $y=1+(x-2)^2$ (ii) $y=2-(x+1)^2$

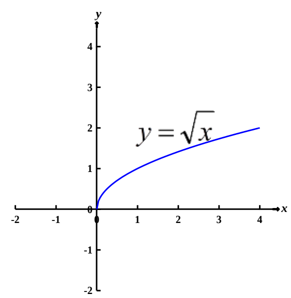
(iii) $y=-2(x+1)^2-3$ (iv) $y=(1/2)(x-3)^2+2$

(v) $y=x^2 + 6x$ (vi) $y=x^2 + 6x -10$

(b) The given function $y = \sqrt{x}$

(i) $y = 3 - \sqrt{x+1}$ (ii) $y = 1 + \sqrt{x-4}$

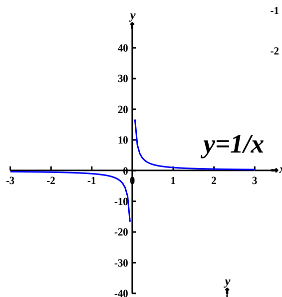
(iii) $y = \frac{1}{2}\sqrt{x} + 1$ (iv) $y = -\sqrt{3x}$



(c) The given function $y = \frac{1}{x}$

(i) $y = \frac{1}{x-3}$ (ii) $y = \frac{1}{1-x}$

(iii) $y = 2 - \frac{1}{x+1}$ (iv) $y = \frac{x-1}{x}$



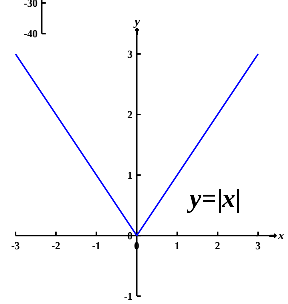
(d) The given function $y=|x|$

(i) $y=|x+2|-2$ (ii) $y=1-|x-3|$

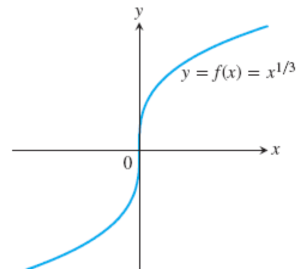
(iv) $y = \sqrt{x^2 - 4x + 4}$

(iii) $y=|2x-1|+2$

$= \sqrt{(x-2)^2} = |x-2|$



(e) The given function $y=\sqrt[3]{x}$



(i) $y=1-2\sqrt[3]{x}$ (ii) $y=\sqrt[3]{x-1}-3$

(iii) $y=2+\sqrt[3]{x+1}$ (iv) $y=-\sqrt[3]{x-2}$

2. Shrink and stretch the following functions along both *x*-axis and *y*-axis by (3/2) units then sketch the resulting function.

(a) $x^2+y^2=4$, $D_f=\{x: -2 \le x \le 2\}$

$R_f=\{y: -2 \le y \le 2\}$

(b) $2x^2+y^2/2=6$, $D_f=\{x: -2 \le x \le 3\}$

$R_f=\{y: -2 \le y \le 2\sqrt{6}\}$

(c) $y=3x^2-2x+1$, $D_f=\{x: -1 \le x \le 2\}$

$R_f=\{y: \frac{6}{9} \le y \le 9\}$